Investigation of all Ricci semi-symmetric and all conformally semi-symmetric spacetimes

Jan E. Åman

Department of Physics, Stockholm University, 106 91 Stockholm

ja@fysik.su.se

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Talk in honor of Brian Edgar (27/12 1945 - 10/6 2010)

Brian Edgar at ERE2007, Teide excursion, Tenerife

Photo:
Narit Pidokrajt

Definition of semi-symmetry

Semi-symmetric spaces where introduced by Cartan 1946 and are characterized by the curvature condition

$$\nabla_{[a}\nabla_{b]}R_{cdef} = 0. \tag{1}$$

This is a generalization of symmetric spaces where $\nabla_b R_{cdef} = 0$.

A semi-Riemannian manifold is said to be *conformally semi-symmetric* if the Weyl tensor C_{abcd} satisfies

$$\nabla_{[a}\nabla_{b]}C_{cdef} = 0; \tag{2}$$

and *Ricci semi-symmetric* if the Ricci tensor R_{cd} satisfies

$$\nabla_{[a}\nabla_{b]}R_{cd} = 0. \tag{3}$$

Spacetimes fulfilling both last conditions are *semi-symmetric*.

Background, Spinors

In this talk all *Ricci semi-symmetric* as well as all *conformally semi-symmetric* spacetimes will be presented. Neither of these properties will imply the other.

Eriksson and Senovilla in *Class. Quantum Grav.* **27** 027001 (2010) [arXiv:0908.3246 [math.DG]] found all **non-conformally flat** *conformally semi-symmetric* spacetimes and pointed out that they all are in fact *semi-symmetric*.

It is an advantage to instead of tensors rather to use the Newman-Penrose spinors, i.e. the Weyl spinor Ψ_{ABCD}, the curvature scalar spinor $\Lambda = \frac{1}{24}R$, and the spinor $\Phi_{ABA'B'}$ for the tracefree part of the Ricci tensor. We also use the spinor $\mathrm{X}_{ABCD} = \Psi_{ABCD} + \Lambda\,(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})$.

Spinor condition for conformal semi-symmetry

The spinor commutator $\square_{AB}$ is operating on spinors with one index defined as $\square_{AB}\,\kappa_C = -\mathrm{X}_{ABC}{}^{E}\kappa_E$ and $\square_{AB}\,\tau_{C'} = -\Phi_{ABC'}{}^{E'}\tau_{E'}$ (Penrose and Rindler 1984 *Spinors and spacetime* vol 1)

In spinors the condition for *conformal semi-symmetry* $\nabla_{[a}\nabla_{b]}C_{cdef} = 0$ is equivalent to $\square_{AB}\Psi_{CDEF} = 0$ and $\square_{A'B'}\Psi_{CDEF} = 0$, or

$$\mathrm{X}_{AB(C}{}^{G}\Psi_{DEF)G} = 0, \qquad (4)$$

$$\Phi_{A'B'(C}{}^{G}\Psi_{DEF)G} = 0. \qquad (5)$$

Calculation of the components of (4) in terms of Ψ_{ABCD} and Λ shows show that it not will have 15 independent components but rather just 5 components and (4) can be replaced by its contraction over BC or

$$\Psi^{GH}{}_{(AD}\Psi_{EF)GH} - 2\Lambda\Psi_{ADEF} = 0. \qquad (6)$$

Spinor condition for Ricci semi-symmetry

We observe that the spinor commutator $\square_{AB}$ yields 0 if operating on a scalar as the curvature scalar Λ, therefore only its effect on $\Phi_{ABA'B'}$, i.e. only the tracefree part of the Ricci tensor has to be considered. The condition for *Ricci semi-symmetry* (3) corresponds to spinor equations

$$\square_{AB}\Phi_{CDC'D'} = -2\mathrm{X}_{AB(C}{}^{E}\Phi_{D)EC'D'} - 2\Phi_{AB(C'}{}^{E'}\Phi_{D')E'CD}, \quad (7)$$

$$\square_{A'B'}\Phi_{CDC'D'} = -2\overline{\mathrm{X}}_{A'B'(C'}{}^{E'}\Phi_{D')E'CD} - 2\Phi_{A'B'(C}{}^{E}\Phi_{D)EC'D'}. \quad (8)$$

The condition (8) is however just the complex conjugate of (7) and will not give any additional conditions for *Ricci semi-symmetry*.

Ricci semi-symmetry expanded in NP-spinors

Equation (7) has three groups of two symmetric indices but will in fact not have 27 independent components as $\Box^{AB}\Phi_{ABC'D'} = 0$, it can be replaced by the fully symmetric spinors

$$\Box_{(AB}\Phi_{CD)C'D'} = -2\Psi_{(ABC}{}^{E}\Phi_{D)EC'D'} \quad (9)$$

$$\Box_{(A}{}^{F}\Phi_{C)FC'D'} = 4\Lambda\Phi_{ACC'D'} - \Psi^{EF}{}_{AC}\Phi_{EFC'D'} - 2\Phi^{E}{}_{A}{}^{F'}{}_{(C'}\Phi_{D')F'CE} \quad (10)$$

with 15 and 9 components respectively.

The conditions for *Ricci semi-symmetry* can now be written as

$$\Psi_{(ABC}{}^{E}\Phi_{D)EC'D'} = 0, \quad (11)$$

$$4\Lambda\Phi_{ACC'D'} - \Psi^{EF}{}_{AC}\Phi_{EFC'D'} - 2\Phi^{E}{}_{A}{}^{F'}{}_{(C'}\Phi_{D')F'CE} = 0. \quad (12)$$

Standard frames, CLASSI

The various spacetimes are for simplicity studied in a frame where first Ψ_{ABCD} has been brought to a standard form depending on its Petrov type. (**0:** $\Psi = 0$, **N:** $\Psi_4 = 1$, **D:** $\Psi_2 \neq 0$, **III:** $\Psi_3 = 1$, **II:** $\Psi_2 \neq 0,\ \Psi_4 = 1$, **I:** $\Psi_2 \neq 0,\ \Psi_0 = \Psi_4 \neq 0$)

Thereafter $\Phi_{ABA'B'}$ is brought to standard form depending on its Segre type. (**A1[(111,1)]:** $\Phi = 0$, **A1[(11)(1,1)]:** $\Phi_{11'} \neq 0$, **A3[(11,2)]:** $\Phi_{22'} \neq 0$, **A1[(111),1]:** $\frac{1}{2}\Phi_{00'} = \Phi_{11'} = \frac{1}{2}\Phi_{22'} \neq 0$, **A1[1(11,1)]:** $-\frac{1}{2}\Phi_{00'} = \Phi_{11'} = -\frac{1}{2}\Phi_{22'} \neq 0$, ...)

All of the above formulas have been implemented in my computer algebra program CLASSI, mainly intended for classification of space-times. All possible combinations of Petrov and Segre types has been examined.

Summary. All semi-symm, conf s-s and Ric s-s spacetimes

Segre type / Petrov type	**I**	**II**	**III**	**D**	**N**	**0**
Λ-term A1[(111,1)] or vacuum	Ric s-s	Ric s-s	Ric s-s	Ric s-s	Ric s-s	semi-sym
Λ-term, $\Lambda = -\frac{1}{2}\Psi_2$	Ric s-s	Ric s-s	$\nexists$	semi-sym	$\nexists$	$\nexists$
A1[(11)(1,1)] ($\Phi_{11'} \neq 0$), $\Lambda = -\frac{1}{2}\Psi_2$	-	-	$\nexists$	semi-sym	$\nexists$	$\nexists$
A1[(11)(1,1)] ($\Phi_{11'} \neq 0$, Λ)	-	-	-	see above	-	semi-sym
A3[(11,2)] ($\Phi_{22'} \neq 0$), $\Lambda = 0$	-	-	-	-	semi-sym	semi-sym
A1[(111),1] perfect fluid, $\Lambda = \frac{1}{2}\Phi_{00'} = \Phi_{11'} = \frac{1}{2}\Phi_{22'}$	-	-	-	-	-	semi-sym
A1[1(11,1)] tachyon fluid, $\Lambda = -\frac{1}{2}\Phi_{00'} = \Phi_{11'} = -\frac{1}{2}\Phi_{22'}$	-	-	-	-	-	semi-sym
All other Ricci tensors	-	-	-	-	-	conf s-s

Table: Relations between Petrov type, Segre type and *conformal semi-symmetry* (conf s-s), *Ricci semi-symmetry* (Ric s-s) and *semi-symmetry*. A hyphen (-) indicates neither *conformal* nor *Ricci semi-symmetry*.

All conformally flat spacetimes are *conformally semi-symmetric*, all spacetimes with Λ-term only are *Ricci semi-symmetric*.

Semi-symmetric spacetimes are flat spacetimes, Λ-term and Segre A1[(11)(1,1)] spacetimes of Petrov type **0** and of type **D** with $\Lambda = -\frac{1}{2}\Psi_2$, Segre type A3[(11,2)] with $\Lambda = 0$ of Petrov types **N** and **0**, as well as conformally flat perfect fluids and tachyons with $\Lambda = \Phi_{11'}$.

Thank you for listening!

Jan E. Åman

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