Reduced phase space quantization of FRW universe

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Motivation

Question: Is the initial singularity of the Universe avoided by quantum gravitational effects?

➡ **We need a quantum theory of the Universe.**

Problem: Time evolution of wave function and observable is lost in quantum gravity.

"Problem of time and observable"

Aims of the work:

- construct a quantum theory free from the problem
- analyze the dynamics of the Universe

Problem of time and observable

What should be interpreted as time and observable in quantum gravity?

Canonical formulation of GR（ADM formulation）

$$S = \frac{1}{\kappa}\int dt \int d^3x \left[\pi^{ab}\dot{q}_{ab} - (N\mathcal{C} + N^a\mathcal{D}_a)\right]$$

Constraints: $\mathcal{C} = 0,\ \mathcal{D}_a = 0$

Hamiltonian: $\mathcal{H} = N\mathcal{C} + N^a\mathcal{D}_a$

Dirac quantization

$$\hat{\mathcal{C}}|\Psi\rangle = 0,\ \hat{\mathcal{D}}_a|\Psi\rangle = 0$$

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{\mathcal{H}}|\Psi\rangle = (N\hat{\mathcal{C}} + N^a\hat{\mathcal{D}}_a)|\Psi\rangle = 0$$

The wave function does not evolve with respect to t.

What is observable?

ADM formulation is a gauge system where the spacetime diffeomorphisms are interpreted as gauge transformations.

⬇

In view of gauge theories,
only gauge-invariant quantities are observables.

O gauge invariant ⇔ $\{\mathcal{C}, O\} = 0, \quad \{\mathcal{D}_a, O\} = 0$

Hamilton's equation :

$$\mathcal{H} = N\mathcal{C} + N^a\mathcal{D}_a$$

$$\dot{O} = \{\mathcal{H}, O\} = N\{\mathcal{C}, O\} + N^a\{\mathcal{D}_a, O\} = 0$$

If one restricts observables to gauge-invariant quantities,
dynamics of observables is lost in classical and quantum gravity.

Flat FRW universe with dust

We use the Ashtekar formulation and consider the dust action introduced by Brown and Kuchar.

- Total action: $S_{\rm tot} = \int dt \left[\frac{3}{\kappa\gamma} p\dot{c} + P_T \dot{T} - N H_{\rm tot} \right]$
  (Gravity + Dust)

  $\left(\kappa = 8\pi G\right.$, γ : Barbero-Immirzi parameter, N: lapse function)

- Gravitational variable: $\{c, p\} = \frac{\kappa\gamma}{3}$

$$|p| = V^{\frac{2}{3}} a^2, \;\; c = \mathrm{sgn}(p) V^{\frac{1}{3}} \frac{\gamma}{N} \dot{a}$$

(a : scale factor, $V = \int d^3x$: 3-volume)

The range of p is the whole real line which is doubled from that of the scale factor a>0. The sign of p determines the orientation of triads.

- Dust variable: $\{T, P_T\} = 1$ (Brown & Kuchar, '95)

T is proper time along flow lines of dust particle.

- Constraint:

$$H_{\rm tot} = H_{\rm grav} + H_{\rm dust} = -\frac{3}{\kappa\gamma^2} c^2 \sqrt{|p|} + P_T = 0$$

Deparametrized form

In the deparametrized case, by using the so-called Relational formalism, one can construct the reduced phase space spanned by gauge-invariant quantities where there are no constraints.

Relational formalism

（Bergmann '61, Rovelli '90, Dittrich '04, Thiemann '06 ）

Relational formalism provides a possible resolution to the problem of time and observable.

Basic idea:

A coordinate time is not a physical time.
Relations between dynamical fields are observable.

Choose the dust variable T as a clock.
Then, the value of a function F at T= τ is gauge invariant.

Definition: $O_F(\tau) := \alpha_C^\lambda(F)|_{\alpha_C^\lambda(T)=\tau}$

α_C^λ : action of the gauge transformation generated by C,
λ is a gauge parameter

Reduced phase space of FRW universe:

Dust variable T serves as a clock.

- Reduced phase space coordinates:

$$C(\tau) := O_c(\tau), \quad P(\tau) := O_p(\tau)$$

The value of p at T=τ.

- Poisson bracket:

$$\{C(\tau), P(\tau)\} = \frac{\kappa\gamma}{3}$$

- Physical Hamiltonian:

$$H_{\text{phys}} = -\frac{3}{\kappa\gamma^2} C(\tau)^2 \sqrt{|P(\tau)|}$$

One-dimensional system with no constraint !!

Quantization

- Commutation relation: $\{C(\tau), P(\tau)\} = \frac{\kappa\gamma}{3}$ ➡ $[\hat{C}, \hat{P}] = \frac{i\kappa\gamma\hbar}{3}$
- Schrodinger representation:

$$\hat{P}\Psi(P) = P\Psi(P), \quad \hat{C}\Psi(P) = \frac{i\hbar\kappa\gamma}{3}\frac{\partial\Psi(P)}{\partial P}$$

- Hamiltonian operator:

$$H_{\mathrm{phys}} = -\frac{3}{\kappa\gamma^2}C(\tau)^2\sqrt{|P(\tau)|} \quad \Longrightarrow \quad \hat{H}_{\mathrm{phys}} = -\frac{3}{\kappa\gamma^2}\sqrt{|\hat{P}|}\hat{C}^2$$

- Hilbert space: $\mathcal{H} = L^2(\mathbb{R}, \underline{|P|^{-\frac{1}{2}}dP})$

The Hamiltonian is self-adjoint in the Hilbert space.

Dynamics of the Universe

Procedure:

1. prepare an initial wave packet at some P
2. numerically evolve it backward in time
3. evaluate the expectation value of |P|

±P correspond to the Universe of the same size with different orientation of triads.

Schrodinger equation:

$$i\hbar\frac{\partial\Psi}{\partial\tau} = \frac{\kappa\hbar^2}{3}\sqrt{|P|}\frac{\partial^2\Psi}{\partial P^2}$$

Initial wave packet:

$$\Psi(P,\tau=0) \propto \exp\left(-\frac{(P-P_0)^2}{4\sigma^2} - ik_0 P\right)$$

Results

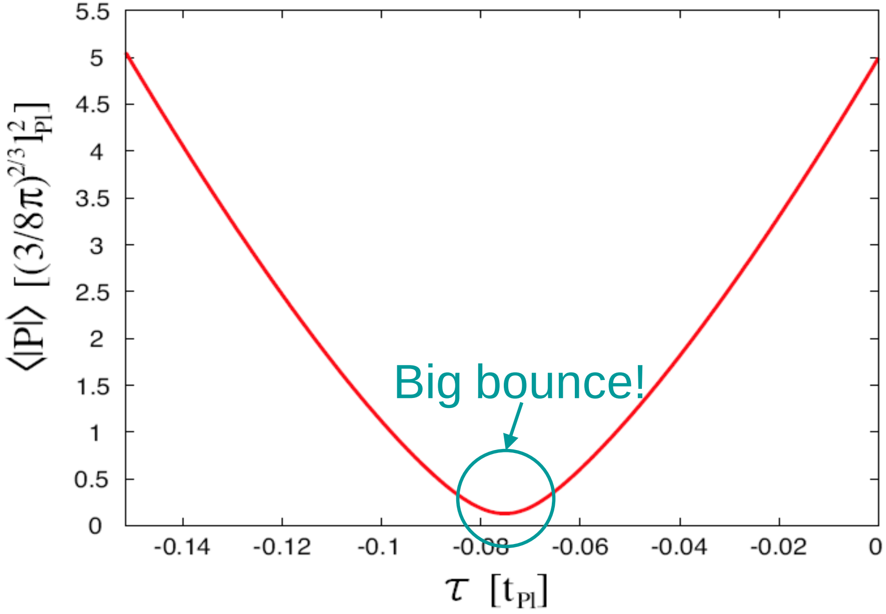

Fig.1 Expectation value of |P|

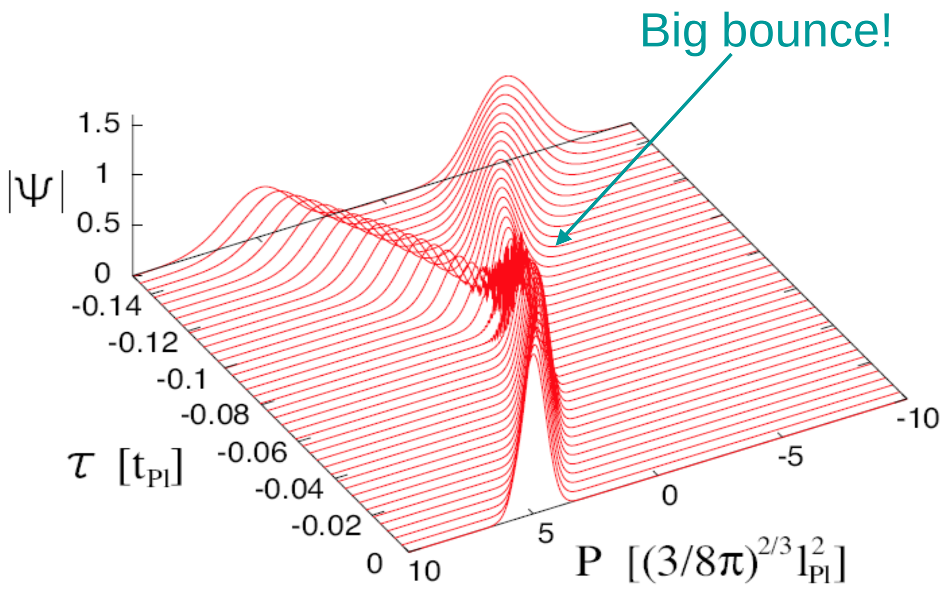

Fig.2 Absolute value of the wave function

- The initial singularity is replaced by a big bounce.
- The universe has been in a superposition of states representing right-handed and left-handed systems before the big bounce.

Summary

- Gauge-invariant construction of quantum cosmology has been proposed.
- The quantization has provided a possible resolution to the problem of time and observable.
- Initial singularity of the Universe has been replaced by a big bounce.
- If the present Universe is in the right-handed state, the past state was in superposition of the right-handed and left-handed states.