The interior of axisymmetric and stationary black holes: **Numerical and analytical studies**

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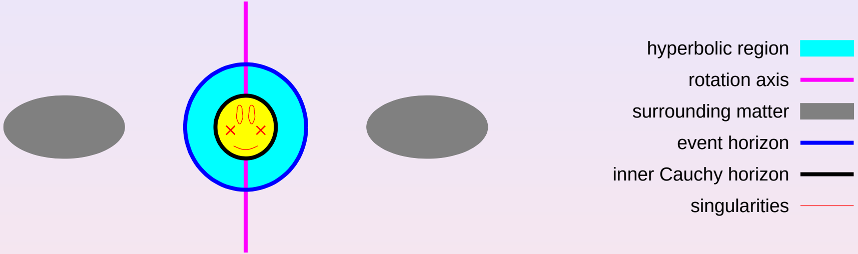
Plan of the talk

The hyperbolic region inside a black hole



- In the hyperbolic region inside the black hole event horizon $\mathcal{H}^+$, any linear combination of the two existing Killing vectors ξ and η yields a space-like vector.
- The axisymmetric and stationary Einstein equations, which are elliptic in the black hole's exterior, become hyperbolic there.

The Kerr solution

- A boundary of the future domain of dependence of the event horizon $\mathcal{H}^+$ can be identified:

  the **inner Cauchy horizon** $\mathcal{H}^-$

- The mathematical form of the field equations at $\mathcal{H}^-$ is completely equivalent to that at $\mathcal{H}^+$.
- Physically, the inner Cauchy horizon is a future horizon whereas the event horizon is a past one.
- The space-time is always regular at $\mathcal{H}^+$.
- It is regular at $\mathcal{H}^-$ only if the black hole's angular momentum J does not vanish.
- For $J \to 0$ the horizon $\mathcal{H}^-$ becomes singular.
- Relation between the areas $A^\pm$ of the two horizons:

$$A^- A^+ = (8\pi J)^2$$

Numerical and Analytical studies

- In this talk we consider general axisymmetric and stationary black holes surrounded by matter and

  1. study the initial value problem of the hyperbolic Einstein equations inside the hyperbolic region.

     We utilize a global **single-domain pseudo-spectral scheme** to find the solution in between and up to the two horizons $\mathcal{H}^{\pm}$.

  2. analyze rigorously the relation between event and inner Cauchy horizon by means of methods from soliton theory.

     We utilize **Bäcklund transformations** in order to express the metric quantities at the inner Cauchy horizon in terms of those at the event horizon.

- **Result:** Proofs of the universal validity of the equality

$$A^{-}A^{+} = (8\pi J)^2$$

Hyperbolic Einstein equations

- Singular Boyer-Lindquist-type line element:

$$ds^2 = \hat{\mu}\left(\frac{dR^2}{R^2 - r_{\mathrm{h}}^2} + d\theta^2\right) + \hat{u}\sin^2\theta(d\varphi^2 - \omega dt)^2 - \frac{4}{\hat{u}}(R^2 - r_{\mathrm{h}}^2)dt^2$$

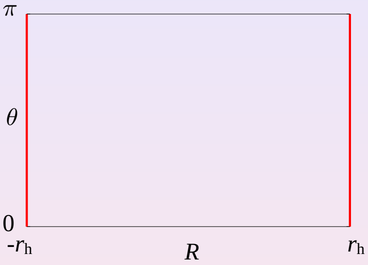
  - $R \in [-r_{\mathrm{h}}, r_{\mathrm{h}}]$, $\theta \in [0, \pi]$
  - Horizons $\mathcal{H}^{\pm} : R = \pm r_{\mathrm{h}}$ Rotation axis: $\theta = 0,\ \theta = \pi$

- Metric coefficients $\hat{\mu}, \omega, \hat{u}$: regular at $\mathcal{H}^{\pm}$
- Hyperbolic Einstein equations: $\left[\tilde{u} = \frac{1}{2}\ln\left(r_{\mathrm{h}}^{-2}\hat{u}\right)\right]$

$$(R^2 - r_{\mathrm{h}}^2)\,\tilde{u}_{,RR} + 2\,R\,\tilde{u}_{,R} + \tilde{u}_{,\theta\theta} + \tilde{u}_{,\theta}\cot\theta = 1 - \frac{\hat{u}^2}{8}\sin^2\theta\left(\omega_{,R}^2 - \frac{\omega_{,\theta}^2}{R^2 - r_{\mathrm{h}}^2}\right)$$

$$(R^2 - r_{\mathrm{h}}^2)\,(\omega_{,RR} + 4\,R\,\omega_{,R}) + \omega_{,\theta\theta} + \omega_{,\theta}(3\cot\theta + 4\tilde{u}_{,\theta}) = 0$$

Analysis of the initial value problem

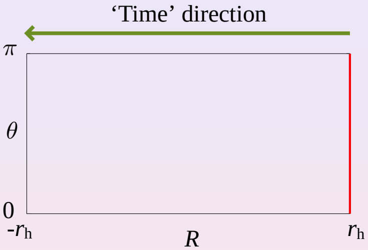


The equations degenerate at the two horizons $\mathcal{H}^{\pm}$.
Consequences for $R = \pm\, r_\mathrm{h}$:

1. $\omega = \mathrm{const.}$
2. $\pm\, 2\, r_\mathrm{h}\, \tilde{u}_{,R} + \tilde{u}_{,\theta\theta} + \tilde{u}_{,\theta} \cot\theta = 1 - \frac{1}{8}\hat{u}^2 \omega_{,R}^2 \sin^2\theta$

Hence: $\tilde{u}_{,R}(\pm\, r_\mathrm{h}, \theta)$ is determined by $\tilde{u}(\pm\, r_\mathrm{h}, \theta)$ and $\omega_{,R}(\pm\, r_\mathrm{h}, \theta)$.

Analysis of the initial value problem



- Ansatz:

$$\tilde{u}(R,\theta) = \tilde{u}_0(\theta) + (R - r_{\mathrm{h}})U(R,\theta)$$
$$\omega(R,\theta) = \omega_0 + (R - r_{\mathrm{h}})\omega_1(\theta) + (R - r_{\mathrm{h}})^2\Omega(R,\theta)$$

- Initial data set: $\{\tilde{u}_0(\theta), \omega_0, \omega_1(\theta)\}$
- Write field equations in terms of the auxiliary functions U and Ω.

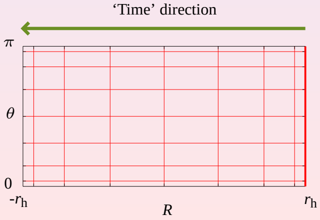
The single-domain pseudo-spectral scheme

- Write functions U and Ω in terms of Chebyshev expansions:

$$U \approx \sum_{j=0}^{n} \sum_{k=0}^{n} c_{jk}^{(U)} T_j\left(\frac{R}{r_{\mathrm{h}}}\right) T_k\left(\frac{2}{\pi}\theta - 1\right)$$

$$\Omega \approx \sum_{j=0}^{n} \sum_{k=0}^{n} c_{jk}^{(\Omega)} T_j\left(\frac{R}{r_{\mathrm{h}}}\right) T_k\left(\frac{2}{\pi}\theta - 1\right)$$

- Consider field equations on a ‘spectral grid’:



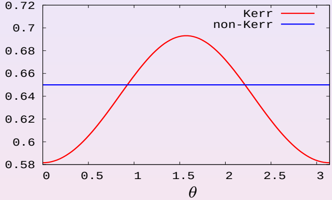
The single-domain pseudo-spectral scheme

- Solve the corresponding discrete non-linear system by means of a Newton-Raphson scheme
- Initial guess taken from Kerr solution for some specific parameters (say M and $a = J/M$).
- Depart from Kerr solution and approach gradually some new solution with non-Kerr initial data set $\{\tilde{u}_0(\theta), \omega_0, \omega_1(\theta)\}$

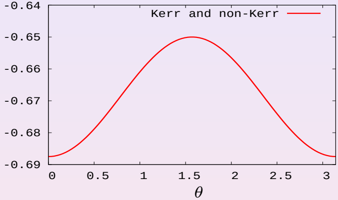
Example: Departure from Kerr with $M = 1$ and $a = 0.8$

Initial data: take $\omega_0 = \omega_0[\mathrm{Kerr}]$ also for non-Kerr example

$\tilde{u}_0$

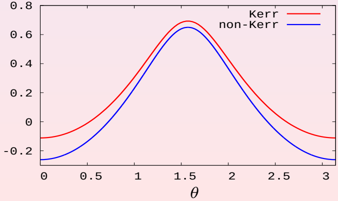


ω_1

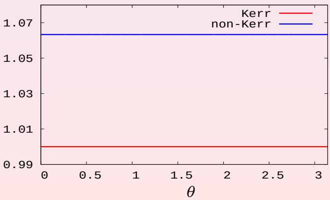


Data at inner Cauchy horizon:

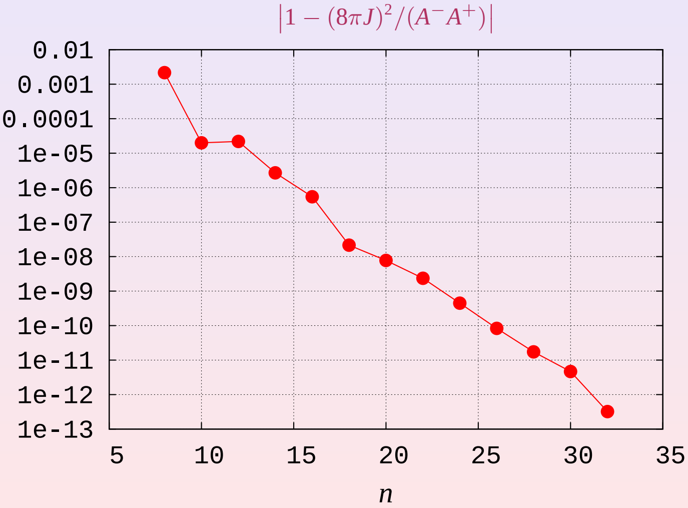
$\tilde{u}$



ω



Example: Convergence plot for non-Kerr solution



Weyl coordinates

- Transition to Weyl coordinates $(\rho, \zeta, \varphi, t)$:

$$\rho^2 = 4(R^2 - r_{\mathrm{h}}^2)\sin^2\theta, \qquad \zeta = 2R\cos\theta.$$

- Line element:

$$\mathrm{d}s^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + \rho^2\mathrm{d}\varphi^2\right] - \mathrm{e}^{2U}(\mathrm{d}t + a\mathrm{d}\varphi)^2$$

  Metric potentials U, k, a are functions of ρ and ζ.
- Along the rotation axis: $\rho = 0, |\zeta| \geq 2r_{\mathrm{h}}$
- $\mathcal{H}^+$ located at $\rho = 0, -2r_{\mathrm{h}} \leq \zeta \leq 2r_{\mathrm{h}}$:
  The event horizon is a degenerate surface in Weyl coordinates.

Weyl coordinates

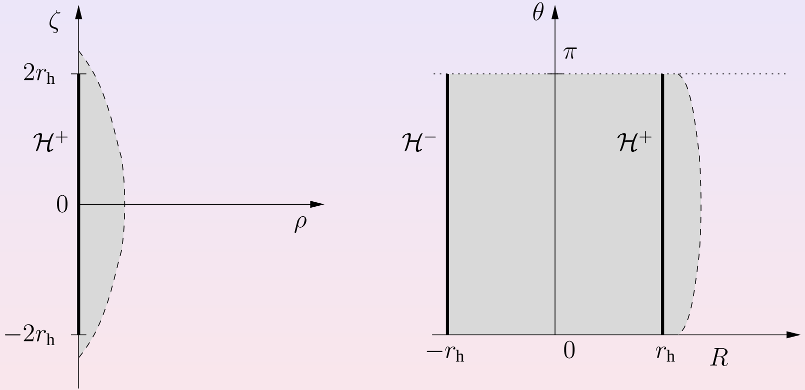


Figure: Portion of a black hole space-time in Weyl coordinates (left panel) and Boyer-Lindquist type coordinates (right panel).

The Ernst equation

- The complex Ernst potential f combines metric functions

$$f = \mathrm{e}^{2U} + \mathrm{i}b.$$

- The twist potential b is related to the coefficient a via

$$a_{,\rho} = \rho\,\mathrm{e}^{-4U} b_{,\zeta}, \qquad a_{,\zeta} = -\rho\,\mathrm{e}^{-4U} b_{,\rho}.$$

- The vacuum Einstein equations are equivalent to the Ernst equation which reads in Weyl coordinates as

$$(\Re f)\left(f_{,\rho\rho} + f_{,\zeta\zeta} + \frac{1}{\rho} f_{,\rho}\right) = f_{,\rho}^2 + f_{,\zeta}^2$$

and in Boyer-Lindquist type coordinates:

$$(\Re f)\left[(R^2 - r_\mathrm{h}^2) f_{,RR} + 2R f_{,R} + f_{,\theta\theta} + \cot\theta f_{,\theta}\right] = (R^2 - r_\mathrm{h}^2) f_{,R}^2 + f_{,\theta}^2.$$

Regularity of the Ernst potential

- Because of the degeneracy of $\mathcal{H}^+$ in Weyl coordinates, the potential f is, for $\rho = 0$, only a C^0-function in terms of ζ.
- However, f is analytic with respect to the Boyer-Lindquist type coordinates R and $\cos\theta$.

Bäcklund transformation

- The Bäcklund transformation is a particular soliton method, which creates a new solution from a previously known one.
- For the Ernst equation this technique can be applied to construct a large number of axisymmetric and stationary space-time metrics.
- We consider the Bäcklund transformation in order to write an arbitrary regular axisymmetric, stationary black hole solution f in terms of a potential f_0.
- Here f_0 describes a space-time without a black hole but with a completely regular central vacuum region.

Bäcklund transformation

Theorem:
Consider a regular axisymmetric and stationary black hole solution f describing a sufficiently small exterior vacuum vicinity V of the event horizon $\mathcal{H}^+$. Then an Ernst potential $f_0 = \mathrm{e}^{2U_0} + \mathrm{i}b_0$ of a space-time without a black hole can be identified with the following properties:

1) f_0 is defined in a vicinity of the axis section $\rho = 0, |\zeta| \le 2r_\mathrm{h}$.
2) In this vicinity, f_0 is an analytic function of ρ and ζ and an **even** function of ρ.
3) The axis values of f_0 in terms of those of f for $\rho = 0, |\zeta| \le 2r_\mathrm{h}$ are given by

$$f_0 = \frac{\mathrm{i}\left[2r_\mathrm{h}(b_\mathrm{N}^+ + b_\mathrm{S}^+) - (b_\mathrm{N}^+ - b_\mathrm{S}^+)\zeta\right] f + 4r_\mathrm{h} b_\mathrm{N}^+ b_\mathrm{S}^+}{4r_\mathrm{h} f - \mathrm{i}\left[2r_\mathrm{h}(b_\mathrm{N}^+ + b_\mathrm{S}^+) + (b_\mathrm{N}^+ - b_\mathrm{S}^+)\zeta\right]},$$

where $b_\mathrm{N}^+ = b(\rho = 0, \zeta = 2r_\mathrm{h})$ and $b_\mathrm{S}^+ = b(\rho = 0, \zeta = -2r_\mathrm{h})$.

Bäcklund transformation

From this Ernst potential f_0 the original potential f can be recovered in all of V by means of an appropriate Bäcklund transformation of the following form:

$$f = \frac{\begin{vmatrix} f_0 & 1 & 1 \\ \bar{f}_0 & \alpha_1\lambda_1 & \alpha_2\lambda_2 \\ f_0 & \lambda_1^2 & \lambda_2^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ -1 & \alpha_1\lambda_1 & \alpha_2\lambda_2 \\ 1 & \lambda_1^2 & \lambda_2^2 \end{vmatrix}},$$

where

$$\lambda_i = \sqrt{\frac{K_i - \mathrm{i}\bar{z}}{K_i + \mathrm{i}z}}, \qquad i = 1, 2, \qquad K_1 = -2r_\mathrm{h}, \qquad K_2 = 2r_\mathrm{h}$$

with the complex coordinates $z = \rho + \mathrm{i}\zeta$, $\bar{z} = \rho - \mathrm{i}\zeta$ and α_1, α_2 are solutions of specific Riccati-type equations.

Deriving f_0 at the interior boundary $R = -r_\mathrm{h}$

- A crucial role is played by the fact that f_0 is **even** in ρ.
- In terms of the Boyer-Lindquist type coordinates, f_0 is an analytic function of $(R^2 - r_\mathrm{h}^2)\sin^2\theta$ and $R\cos\theta$.
- The analytic expansion of f_0 into the region $R < r_\mathrm{h}$ retains this property.
- Hence: f_0, taken at the boundaries of the inner hyperbolic region, can be expressed in terms of f_0 taken at $R = r_\mathrm{h}$.

  Specifically:

$$f_0(R = \, -r_\mathrm{h}\,, \cos\theta) = f_0(R = \, +r_\mathrm{h}\,, -\cos\theta)$$

- From the values of f_0 at these boundaries we can construct f on $\mathcal{H}^-$ via the Bäcklund transformation.

The Ernst potential on the Cauchy horizon (1)

Theorem:

1) Any Ernst potential f of a regular axisymmetric and stationary black hole space-time with angular momentum $J \neq 0$ can be regularly extended into the interior of the black hole up to and including an interior Cauchy horizon, described by $R = -r_\mathrm{h}$ in Boyer-Lindquist type coordinates (R, θ).

The Ernst potential on the Cauchy horizon (2)

2) The Ernst potential on the Cauchy horizon is given by

$$f(R=-r_{\mathrm{h}},\cos\theta)=\frac{\mathrm{i}\left[\delta_1+\delta_2-(\delta_1-\delta_2)\cos\theta\right]f_0(R=r_{\mathrm{h}},-\cos\theta)+2\delta_1\delta_2}{2f_0(R=r_{\mathrm{h}},-\cos\theta)-\mathrm{i}\left[\delta_1+\delta_2+(\delta_1-\delta_2)\cos\theta\right]}$$

with

$$\delta_1=\frac{b_{\mathrm{S}}^{+}(b_{\mathrm{N}}^{+}-b_{\mathrm{S}}^{+})+2b_{\mathrm{N}}^{+}(b_{,\theta\theta})_{\mathrm{N}}^{+}}{b_{\mathrm{N}}^{+}-b_{\mathrm{S}}^{+}+2(b_{,\theta\theta})_{\mathrm{N}}^{+}},$$

$$\delta_2=\frac{b_{\mathrm{N}}^{+}(b_{\mathrm{N}}^{+}-b_{\mathrm{S}}^{+})+2b_{\mathrm{S}}^{+}(b_{,\theta\theta})_{\mathrm{N}}^{+}}{b_{\mathrm{N}}^{+}-b_{\mathrm{S}}^{+}+2(b_{,\theta\theta})_{\mathrm{N}}^{+}}$$

The scripts '+' and 'N/S' indicate that the corresponding value of b or its second θ-derivative has to be taken at the event horizon's north or south pole respectively.

The Ernst potential on the Cauchy horizon (3)

- The values of the seed solution f_0 for $R = r_\mathrm{h}$ follow from f on the event horizon,

$$f_0 = \frac{\mathrm{i}\left[2r_\mathrm{h}(b_\mathrm{N}^+ + b_\mathrm{S}^+) - (b_\mathrm{N}^+ - b_\mathrm{S}^+)\zeta\right] f + 4r_\mathrm{h} b_\mathrm{N}^+ b_\mathrm{S}^+}{4r_\mathrm{h} f - \mathrm{i}\left[2r_\mathrm{h}(b_\mathrm{N}^+ + b_\mathrm{S}^+) + (b_\mathrm{N}^+ - b_\mathrm{S}^+)\zeta\right]}$$

- For $J \to 0$ the Cauchy horizon becomes singular.

A universal equality

Theorem:

Every regular axisymmetric and stationary black hole with non-vanishing angular momentum J satisfies the relation

$$(8\pi J)^2 = A^+ A^-$$

where $A^\pm$ are the horizon areas of event horizon ($\mathcal{H}^+$) and inner Cauchy horizon ($\mathcal{H}^-$).

- The relation between the two horizons emerges from the fact that they are connected by the symmetry axis.
- Generalization in Einstein-Maxwell theory describing an axisymmetric and stationary black hole with electric charge Q:

$$(8\pi J)^2 + (4\pi Q^2)^2 = A^+ A^-$$

- For subextremal black holes (which possess trapped surfaces in every sufficiently small interior vicinity of $\mathcal{H}^+$), the following inequalities hold:

$$A^- < \sqrt{(8\pi J)^2 + (4\pi Q^2)^2} < A^+$$