Black hole entropy: lessons from loop quantum gravity

J. Fernando Barbero G.
Instituto de Estructura de la Materia, CSIC.
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Introduction and motivation

This morning theme:

Fundamental vs. Effective Approaches in Theoretical Gravity

How does my talk fit into this scheme?

- Any **fundamental** approach to quantum gravity should be able to explain the **origin and meaning** of the black hole microscopic degrees of freedom. (Also describe the appearance of objects that behave like macroscopic black holes!)

The purpose of this talk is to

1. Explain how LQG describes the black hole d.o.f.'s
2. Argue that LQG provides a good model for BH entropy.
3. Show that LQG makes some intriguing predictions.

A little bit of history

- The study of black hole entropy is a **natural testing ground** for LQG
- **Some key steps:**
  - **Smolin** ('95): black holes as inner boundaries in LQG, horizons and Chern-Simons theories, quantum groups,...
  - **Krasnov** ('96-98): role of CS, spin networks, distinguishability,...
  - **Rovelli** ('96): combinatorial methods within LQG, distinguishability of punctures, Bekenstein-Hawking law,...
  - **Ashtekar, Baez, Corichi and Krasnov** ('98-00): BH's as isolated horizons, Hamiltonian formulation, role of γ,...
  - **Domagala, Lewandowski** and **Meissner** ('04): reformulation of combinatorial problems, precise counting,...
  - **Corichi, Díaz Polo, F-Borja** ('07): Entropy quantization.

Circa 1998 there was a reasonable description of BH's within LQG.

- **Recent advances.**

- **Non expanding horizons** Δ: null, 3-dim submanifolds of $(\mathcal{M}, g_{ab})$ such that
    1. They have the topology of $S^2 \times \mathbb{R}$.
    2. The expansion of any null normal $q^{ab}\nabla_a \ell_b$ vanishes.
    3. The Einstein field equations hold on Δ with T_{ab} satisfying the (mild) condition that $-T^a{}_b \ell^b$ is a future directed, causal vector. (This is true for minimally coupled matter fields.)
- This definition (2) guarantees that the **area** of Δ **is constant**, there is **no matter flux** through Δ, and the horizon geometry $(q_{ab}, \mathcal{D})$ is time-independent.
- **Isolated horizons**: Non-expanding horizons with an –essentially unique– null normal ℓ that **is a symmetry** of the horizon geometry ($\mathcal{L}_\ell q_{ab} = 0\,, [\mathcal{L}_\ell, \mathcal{D}] = 0$). It is a much **weaker concept than that of a Killing horizon** (just the minimum necessary have the laws of black hole mechanics and keep an infinite number of d.o.f.) In particular it is a **local concept** (unlike event horizons).

1. **Isolated horizons** capture important features of BH physics (for instance, $0^{\rm th}$ and $1^{\rm st}$ laws of BH thermodynamics).
   - $0^{\rm th}$ law: The surface gravity κ of an isolated horizon is constant.
   - $1^{\rm st}$ law: "Conservation of energy"

$$\mathrm{d}E_\Delta = \frac{\kappa}{8\pi G}\mathrm{d}a_\Delta + \Omega \mathrm{d}J_\Delta + \Phi \mathrm{d}Q_\Delta$$

2. They are appropriate to model black holes in **equilibrium** without requiring that the *exterior geometry* be stationary.
3. They can model rotating black holes or black holes with distorted horizons.
4. They have interesting classical applications.

4. A "reduction" of general relativity consisting of spacetimes with isolated horizons as inner boundaries admits a **Hamiltonian description**. This is a **key first step** towards quantization.
5. This idea is **similar in spirit** to the study of the quantization of mini and midisuperspace models. A subset of the gravitational field configurations is selected by imposing restrictions on the metrics.
   - Mini and midisuperspaces $\rightsquigarrow$ **Symmetry** requirement on the metrics.
   - Black holes $\rightsquigarrow$ The allowed metrics must **have an isolated horizon** that is also an inner boundary of spacetime.
6. **According to the symmetries** of the isolated horizon geometry they are classified as
   - Type I: maximal symmetry corresponding to spherical geometry.
   - Type II: 2-dim symmetry group. Axi-symmetric
   - Type III: 1-dim symmetry group. The only symmetries are the diffeomorphisms generated by the null normal ℓ.
7. The 0^{th} and 1^{st} laws are satisfied by types I and II.

1. Consider a sector of GR consisting of space times with an inner boundary which is a **type I** isolated horizon of fixed classical area a_κ
   - Use real Ashtekar variables (A_a^i, E_i^a) ($SU(2)$ connection and triad).
   - Partially gauge fix $SU(2)$ to $U(1)$ by projecting in the internal direction given by the (internal) vector r^i, where $r^i E_i^a = \sqrt{|\det q|} r^a$ (r^a is the unit normal to the 2-sphere S.)
   - The intrinsic geometry of Δ is given by the pull-back of $A^i r_i =: W$ to S.
2. Δ isolated horizon has two important consequences:
   1. The **boundary condition** (Σ^i is the pull-back to S of $\eta_{abc} E_i^a \eta^{ij}$ and $F := dW$). It implies the reducibility of the pulled back connection

$$F = -\frac{2\pi}{a_\kappa} 8\pi G \gamma \Sigma^i r_i$$

   2. The symplectic form picks a $U(1)$ Chern-Simons surface term

$$\frac{1}{2\pi} \frac{a_\kappa}{4\pi\gamma\ell_P^2} \oint d_1 W \wedge d_2 W$$

3. Constraints

Steps to quantization

1. Build the Hilbert space (describing both of surface and bulk quantum d.o.f.)
   - Bulk Hilbert space
   - Surface Hilbert space
2. Implement the **quantum boundary condition** (**qbc**) to define $\mathcal{H}_{kin}$.
3. Enforce the quantum constraints (*à la Dirac*, group averaging,...) to define $\mathcal{H}_{phys}$.
   - Gauss law.
   - Diffeomorphism constraint.
   - Hamiltonian constraint.

Define then the entropy and study its properties (dependence on the horizon area...)

The Hilbert space

1. We have two types of connections: a **bulk** $SU(2)$ connection and a **surface** $U(1)$ connection
2. It is then natural to take a Hilbert space $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ and consider a kinematical subspace $\mathcal{H}_{kin} < \mathcal{H}_V \otimes \mathcal{H}_S$.
3. The details of the choice for $\mathcal{H}_V$ and $\mathcal{H}_S$ are dictated by the need to implement a **quantum boundary condition**.

Volume Hilbert space $\mathcal{H}_V$

- $\mathcal{H}_V$ is a subspace of the LQG Hilbert space $L^2(\bar{\mathcal{A}}, \mu_{AL})$ (we mode out SU(2) gauge transformations that reduce to $\mathbb{I}$ on the horizon).
- A suitable basis is defined by using **spin networks** (that may end at S).Remember that spin networks are graphs with **edges** consisting of analytic curves that meet at **vertices**. Edges are labeled by **spins** j_e and vertices by **intertwiners** i_v.

- Those spin networks with graphs piercing the horizon transversally (at the so called **punctures**) play an special role. Important quantum numbers are the $j_I \in \mathbb{Z}/2$ labels of their edges and the $//m_I = -j_I, \ldots, j_I$ components defined by the r_i vector.
- The bulk Hilbert space admits a convenient representation as an orthogonal sum:
  - Consider a finite set of points on S, $P = \{P_1, P_2, \ldots, P_n\}$ labeled by pairs of numbers (j_i, m_i), $i = 1, \ldots, n$ such that $j_i \in \mathbb{N}/2$ and $m_i \in \{-j_i, -j_i + 1, \ldots, j_i\}$

$$\mathcal{H}_V = \bigoplus_{(P,j,m)} \mathcal{H}_V^{P,j,m}$$

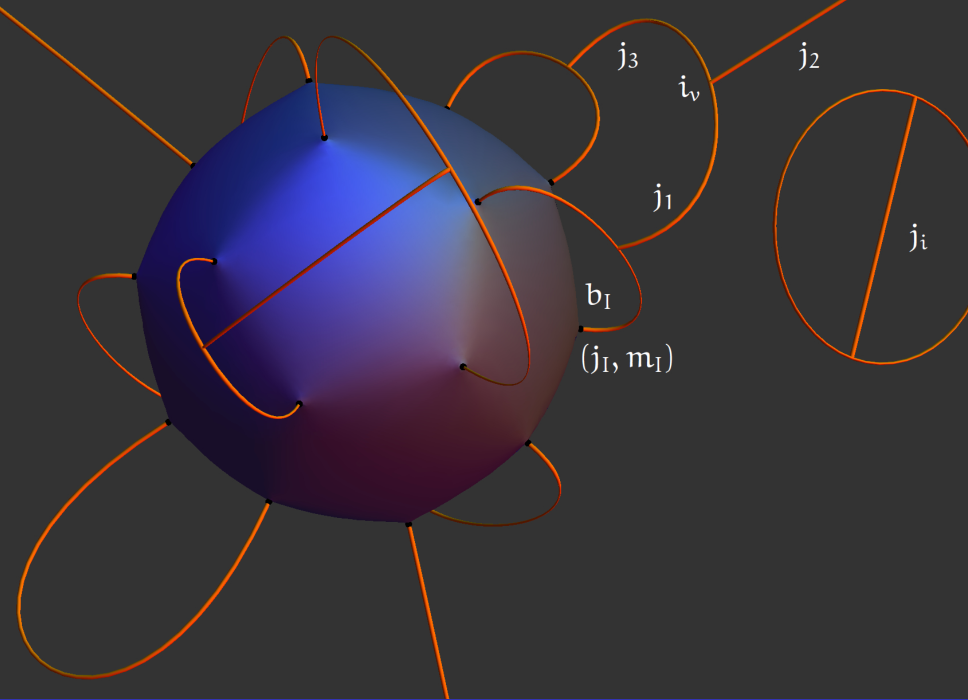
  - The sum is extended to all the possible subsets P (including $\emptyset$) and (j, m)-labelings. $\mathcal{H}^\emptyset \rightsquigarrow$ spin networks that do not pierce the horizon.
  - Each of the Hilbert spaces $\mathcal{H}_V^{P,j,m}$ is spanned by spin networks such that the edges piercing the horizon carry the labels (j_i, m_i).

Surface Hilbert space $\mathcal{H}_S$

- The appearance of a $U(1)$ Chern-Simons surface term in the symplectic structure forces us to impose a "pre-quantization condition" on the classical horizon area $a_\kappa = 4\pi\gamma\ell_P^2\kappa$ ($\kappa \in \mathbb{N}$) for consistency if we take the Hilbert space of the $U(1)$ Chern-Simon theory defined on a punctured sphere.
- $\mathcal{H}_S$ admits a representation as an orthogonal sum:
  - Consider a finite sequence of points on S, $\vec{P} = (P_1, P_2, \ldots, P_n)$ labeled by numbers $b_i \in \mathbb{Z}_\kappa$, $i = 1, \ldots, n$ satisfying $b_1 + \cdots + b_n = 0$.

$$\mathcal{H}_S = \bigoplus_{(\vec{P},b)} \mathcal{H}_S^{\vec{P},b}$$

  - The sum is extended to all the possible sequences $\vec{P}$ and b-labelings. (we introduce a 1-dim Hilbert space $\mathcal{H}_S^{\vec{\emptyset}}$ for the empty sequence.)



j_3
i_v
j_2
j_1
j_i
b_I
(j_I, m_I)

The Kinematical Hilbert space $\mathcal{H}_{kin}$

- We have to implement the quantum boundary condition in $\mathcal{H}_V \otimes \mathcal{H}_S$ (to take into account that S is a section of an isolated horizon Δ.)

$$(\mathbb{I} \otimes \underbrace{e^{i\int_R \hat{F}}}_{\text{holonomies}})\Psi = \Big(\underbrace{e^{-\frac{2\pi i}{a_\kappa}8\pi G\gamma\int_R(\hat{\Sigma}\cdot r)}}_{\text{fluxes}} \otimes \mathbb{I}\Big)\Psi$$

- The solutions are orthogonal sums of $\mathcal{H}_V^{P,j,m} \otimes \mathcal{H}_S^{\vec{P},b}$ such that
  1. The points in $P = \{P_1, \dots, P_n\}$ coincide with the elements in the sequence $(P_1, \dots, P_n)$.
  2. $b_i = -2m_i \,(\mathrm{mod}\kappa)$ for $i = 1, \dots, n$.

$$\mathcal{H}_{kin} = \bigoplus_{\vec{P},j,m} \mathcal{H}_V^{P,j,m} \otimes \mathcal{H}_S^{\vec{P},b(m)}$$

1. The eigenvalues of the lhs and rhs of the qbc can be matched **precisely** if a_κ is the pre-quantized value of the area introduced above $\rightsquigarrow$ non trivial space of solutions to the quantum matching conditions.
2. **Classically** the bulk fields determine the surface fields by **continuity**.
3. **Quantum mechanically**, distributional objects appear so there is no continuity requirement. (Quantum configuration $\neq$ classical configuration space.) This explains the appearance of independent surface d.o.f. accounting for the BH entropy (**a genuine quantum effect!**)
4. There are two areas that play a role here:
   - The classical prequantized area a_κ fixed for the isolated horizon.
   - The area eigenvalue of the bulk **area operator** associated to the 2-sphere S defining the horizon given by

$$Area_S = 8\pi\gamma\ell_P^2 \sum_{I=\text{punct.}} \sqrt{j_I(j_I+1)}$$

   - Notice that spin networks are eigenstates of the area operator.

The physical states of $\mathcal{H}_{phys}$ are solutions to the quantized constraints:

1. **Gauss law:** The physical states must be $SU(2)$ invariant.
2. **Diffeomorphism constraint:** Diffeos act non-trivially on the surface and volume states but the full physical state is invariant. The only thing that matters is the number of punctures but not their location.
3. **Hamiltonian constraint:** It does not restrict the surface states because functional differentiability of this constraint in the classical theory demands that the lapse N is zero on S. (However, one has to assume the existence of *some* solutions to it in $\mathcal{H}_V$ for every choice of (j, m)).

$$\mathcal{H}_{phys} = \bigoplus_{j,m} \mathcal{H}^{b(m),j,m}$$

The area assigned to S by the bulk geometry is controlled by the j's.
The intrinsic horizon degrees of freedom are represented by the b's.

- Consider the Hilbert space of "black hole states"

$$|(m_1, j_1, \ldots, m_n, j_n), \text{the rest of the graph}\rangle_V \otimes |(b_1, \ldots, b_n)\rangle_S, \qquad n \in \mathbb{N}_0 \,,$$

with $b_i \equiv -2m_i \pmod{\kappa}$, for $i = 1, \ldots, n$.

- Introduce an area interval $[a_\kappa - \delta, a_\kappa + \delta]$ with a δ of the order of ℓ_P^2.
- The entropy is computed by tracing out the bulk degrees of freedom to get a density matrix that describes a **maximal entropy mixture** of surface states with area eigenvalues in $[a_\kappa - \delta, a_\kappa + \delta]$.
- This amounts to counting the number of **allowed lists** $(b_1, \ldots, b_n)$ of non-zero elements of $\mathbb{Z}_\kappa$ satisfying $b_1 + \cdots + b_n = 0$, such that $b_i = -2m_i \pmod{\kappa}$ for some *permissible* list of spin components $\vec{m}$.
- Here *permissible* means that there exists a list of non-vanishing spins $(j_1, \ldots, j_n)$ such that each m_i is a spin component of j_i and

$$a_\kappa - \delta \leq 8\pi\gamma\ell_P^2 \sum_{i=1}^{n} \sqrt{j_i(j_i+1)} \leq a_\kappa + \delta.$$

ANOTHER WAY TO GET BLACK HOLE ENTROPY (Domagala-Lewandowski)

The entropy S of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework is

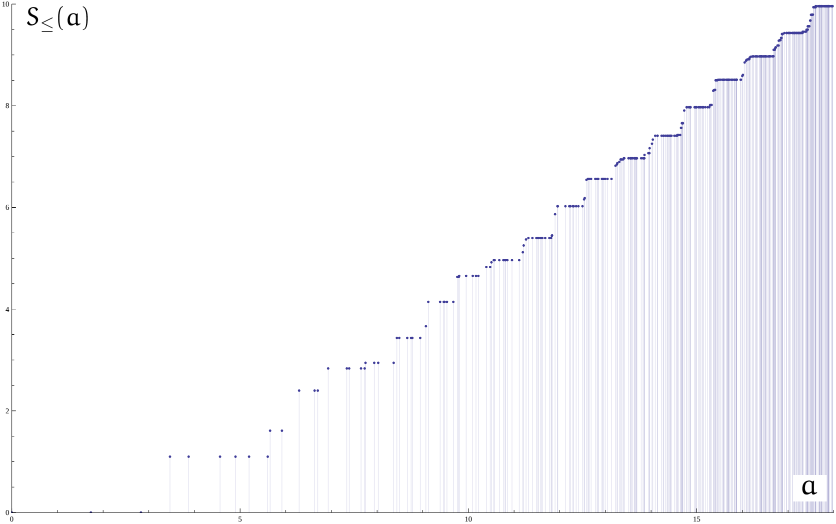
$$S_{\leq}(a) = \log \mathfrak{n}(a),$$

where $\mathfrak{n}(a)$ is 1 plus the number of all finite sequences $\vec{m} := (m_1, \ldots, m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality (the projection constraint) and inequality are satisfied:

$$\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{8\pi\gamma\ell_P^2}$$

where γ is the Immirzi parameter of Quantum Geometry.

(Easier to use in practice.) It extends the entropy to arbitrary values of the area. Let us see what we get if we plot entropy versus area...



in units of $4\pi\gamma\ell_P^2$

PLOTTING ENTROPY VERSUS AREA

1. The entropy is plotted for each area eigenvalue for small black holes. For some of them it is zero because there are no solutions to the projection constraint.
2. The expected **linear growth** as a function of the area is observed (see the classical papers on the subject!)
3. The Immirzi parameter $\gamma = 0.274\cdots$ has to be **chosen** to get the $1/4$ coefficient in the area law (the slope is controlled by γ).
4. However, this choice is **universal** (it works for other types of black holes modeled as type II isolated horizons (rotating and distorted)), Maxwell fields, dilatonic black holes, non minimally coupled matter,...
5. An unexpected behavior appears! It looks as if there is an **effective entropy quantization** (Corichi, Díaz Polo, F. Borja, Agulló). This is a **genuinely new phenomenon**!)
6. The step size remains constant despite the fact that the separation between consecutive area eigenvalues shrinks very quickly!

What does this mean?
Is it possible to explain this new behavior?
Does it persist for macroscopic black holes?

The nature of the combinatorial problems (in particular in the DL form) is such that it is possible to develop counting methods to:

1. Characterize the degeneracies in the area eigenvalues by using diophantine equations (the **Pell equation** $x^2 - py^2 = 1$ appears due to the terms involving $\sqrt{j(j+1)}$ in the area eigenvalues.)
2. **Precisely** obtain the number of black hole configurations contributing to the entropy for each value of the black hole area (this means solving an inequality and the projection constraint).

Furthermore, it is possible to code this information in **generating functions** and use them to study the **asymptotic behavior** of the entropy.

Generating functions

- The **black hole generating function** is

$$G(z,x_1,x_2,\dots)=\left(1-\sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}(z^{k_\alpha^i}+z^{-k_\alpha^i})x_i^{y_\alpha^i}\right)^{-1}$$

- In the previous formula the variables x_i are associated to **squarefree integers** p_i and z is an extra variable needed to account for the projection constraint.
- The numbers (k_α^i, y_α^i) are obtained from the solutions to the **Pell equation** $x^2-p_iy^2=1$ associated to the squarefree integer p_i.
- The coefficient of the term $z^0x_1^{q_1}\cdots x_i^{q_i}\cdots$ gives the number of sequences $\vec{m}$ satisfying the projection constraint and such that $2a=\sum_i q_i\sqrt{p_i}$ (in units such that $4\pi\gamma\ell_P^2=1$)
- Despite the apparent infinite number of terms, for a given value of a **only finite numbers** of variables and terms are needed.

Generating functions

- This generating function solves the problem of counting the number of sequences $\vec{m}$ such that

$$2\sum_{I}\sqrt{|m_I|(|m_I|+1)}=a\,,\quad \sum_{I} m_I = 0$$

for a given value of the area a.

- In order to **take into account the inequality** in the definition of the area one has to introduce a Laplace-Fourier transform representation.

$$e^{S_{\leq}(a)}=\frac{1}{(2\pi)^2 i}\int_0^{2\pi}\int_{x_0-i\infty}^{x_0+i\infty} s^{-1}\Big(1-2\sum_{k=1}^{\infty}e^{-s\sqrt{k(k+2)}}\cos\omega k\Big)^{-1}e^{as}\,\mathrm{d}s\,\mathrm{d}\omega$$

This is an **exact** expression for the entropy. It allows, in particular, to write down the equation that γ must satisfy to recover the 1/4 factor of the Bekenstein-Hawking law.

Generating functions

- The position of the poles of the integrand for $\omega = 0$ determines the growth of the entropy as a function of the area (the one with the largest real part). **However**, a very interesting phenomenon happens: the **real parts of these poles accumulate**.
- This is a necessary condition to eventually explain the entropy quantization for large areas.
- The formula gives a (γ-independent!) **logarithmic correction**

$$-\frac{1}{2}\log\frac{a}{\ell_P^2}$$

- By carefully modifying the generating functions it is possible to isolate parts of the black hole degeneracy spectrum (steps). An example:

$$\left(1-\sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}\nu^{3k_\alpha^i+2}(z^{k_\alpha^i}+z^{-k_\alpha^i})x_i^{y_\alpha^i}\right)^{-1}$$

Some other approaches to understand BH entropy

- Euclidean methods and partition functions (Gibbons, Hawking).
- Entanglement entropy between degrees of freedom inside and outside the horizon (Bombelli, Srednicki, Sorkin).
- Black hole entropy is a conserved quantity connected with the diffeomorphism invariance of the gravitational action. (Wald)
- Black hole entropy is thermal entropy of the gas of quanta constituting the thermal atmosphere of the black hole. (Thorne and Zurek, 't Hooft). This is related to the entanglement entropy.
- Black hole entropy counts the number of states or excitations of a fundamental string. (Strominger and Vafa [1996], Susskind [1993], Bowick, Smolin and Wijewardhana [1987]).
- Black hole entropy is equivalent to the thermal entropy of the radiation residing on the boundary of the spacetime containing the black hole. (Maldacena, Witten).

Confronting other approaches

- Shape of the horizon (Sorkin, 1996).
- Entropy from non-local field theories (Padmanabhan).
- Causal sets (Dou) 1999
- Sakharov's theory of induced gravity (the dynamical aspects of gravity arise from the collective excitations of massive fields with constraints introduced to cancel divergences and ensure that $\Lambda = 0$.
- Mukhanov, Bekenstein,...
- ...

Strings

- At low energies they reduce to a 10-dim SUGRA theory (that involves a metric and can have black holes).
- One can consider a weak coupling limit of string theory where perturbative computations can be performed.
- There are ways to map weak coupling states with black hole solutions (because there are conserved charges if D-branes are introduced).
- In the weak coupling limit one can then use methods of standard statistical mechanics to get the entropy and then map.
- For certain classes of **extremal** (or **almost extremal**) black holes one gets $S(A) = A/4$ and subleading corrections that agree with the Wald formula for GR extensions. **Greybody factors** can be obtained.
- There is a **correspondence principle** put forward by Horowitz and Polchinski that leads to the proportionality of entropy and areas for **generic** black holes (but does not give the 1/4 of the area law).
- There is progress in computing the entropy for "more physical" black holes (Horowitz, Emparan, Roberts,...).

Other approaches and points of view

- **The Universality enigma** (Carlip): "The Bekenstein-Hawking entropy can be computed entirely within the framework of quantum field theory in a fixed curved background. It is hard to see how such a calculation could "know" the details of a microscopic gravitational theory. Rather, it seems more likely that some unknown universal mechanism forces any suitable quantum theory to give the standard result"

- Strominger observed that the **Cardy formula** can be used to compute the entropy and get the Bekenstein-Hawking formula (2+1 black holes).
- Carlip has generalized this approach for **arbitrary black holes** by carefully looking at the symmetries of the horizons.
- The Cardy formula [that comes from 2-dimensional conformal field theory] gives the correct Bekenstein-Hawking entropy independent of the details of the black hole considered.

Other approaches and points of view

- **Logarithmic corrections to the entropy** are sensitive to details of the formulation (stressed by many authors in different contexts and suggested by Kaul, Majumdar, Engle, Noui,Perez,...)
- Conformal field theory arguments can explain the "universality". In particular Carlip has found that the logarithmic correction

$$-\frac{3}{2}\log\frac{a}{\ell_P^2}$$

appears generically. (String theories give this type of correction.)
- *"Microscopic states responsible for black hole entropy can thus be viewed as "would-be pure gauge" states that become physical because the symmetry is altered by the requirement that a horizon exists"*.

Summary

Black hole entropy in LQG

- BH's in LQG are modeled with the help of **isolated horizons**.
- A **Hamiltonian description** is available $\rightsquigarrow$ quantization.
- Quantization is performed in a LQG-like Hilbert space.
- An important role is played by a CS theory defined on the horizon.
- BH entropy is defined according to the standard rules of quantum statistical mechanics (density matrix).

Main results about black hole entropy

- The **Bekenstein-Hawking** law is recovered.
- **Logarithmic corrections** can be explicitly obtained and understood.
- Interesting **details** in the behavior of the entropy $S(A)$ as a function of the horizon area. Suggestive connections with
  - Bekenstein-Mukhanov proposals (equally spaced area spectrum).
  - Conformal theories (Carlip, Kaul-Majumdar,...)
  - ...

Conclusions

- The identification of the microscopic BH dof in LQG relies on a very specific proposal to define the sector of the theory that one quantizes (by using **isolated horizons**).
- It makes use of techniques introduced in LQG and some features of the full dynamics of GR but it is quite generic (independent of many dynamical details).
- A central issue is the identification of the **quantum boundary condition** and its non-trivial quantization.
- **Quantum geometry** plays an important role (area operator).
- The **combinatorial problems** associated with the counting of bh states can be solved in detail. In particular the detailed behavior of the entropy as a function of the area can be studied.
- Even if you consider the present approach to BH entropy in LQG as a model (that can be improved once the complete theory is available) in my opinion it is **too rich to be dismissed**. It may well be the basis for a deeper understanding of black hole physics.