Noncommutative Black Holes

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ERE 2010, Granada

6th-10th September 2010

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Motivation

- Black Holes (BHs) radiate Thermodynamics
  - Quantum Gravity
  - Minisuperspace approximation (Quantum Cosmology)
- Noncommutative Space-Time (NC):
  - String / M-Theory
  - Gravitational Quantum Well
  - Putative signature of Quantum Gravity

Use a phase-space NC genelarization of the Kantowski-Sachs cosmological model to examine the interior of a Schwarzchild BH.

Calculate thermodynamical properties of a Schwarzschild BH and study its singularity.

Schwarzschild vs Kantowski-Sachs:

General Relativity solutions where the causal structure of space-time changes at different regions of space-time.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- r<2M, time and radial coordinates interchange.

$$ds^2 = -\left(\frac{2M}{t}-1\right)^{-1}dt^2 + \left(\frac{2M}{t}-1\right)dr^2 + t^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

- An isotropic metric turns into an anisotropic one
- Mapped to the Kantowski-Sachs metric

$$ds^2 = -N^2dt^2 + e^{2\sqrt{3}\beta}dr^2 + e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega}(d\theta^2 + \sin^2\theta d\varphi^2)$$

- Away from the horizon t=r=2M:

$$N^2 = \left(\frac{2M}{t}-1\right)^{-1} \quad , \quad e^{2\sqrt{3}\beta} = \left(\frac{2M}{t}-1\right) \quad , \quad e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega} = t^2$$

Phase Space Noncommutative Extension of Quantum Mechanics:

$$[\hat{q}_i, \hat{q}_j] = i\theta_{ij} \quad , \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad , \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij} \quad , \quad i, j = 1, ..., d$$

- θ_{ij} e η_{ij} antisymmetric real constant (*dxd*) matrices

- Seiberg-Witten map: class of non-canonical linear transformations
  - Relates standard Heisenberg algebra with noncommutative algebra

- States of system:
  - wave functions of the ordinary Hilbert space

- Schrödinger equation:
  - Modified θ,η-dependent Hamiltonian
  - Dynamics of the system

Noncommutative Quantum Cosmology:

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} (d\theta^2 + \sin^2\theta d\varphi^2)$$

- □, □: scale factors, N: lapse function
- ADM Formalism → Hamiltonian for KS metric:

$$H = N\mathcal{H} = N e^{\sqrt{3}\beta + 2\sqrt{3}\Omega} \left[-\frac{P_\Omega^2}{24} + \frac{P_\beta^2}{24} - 2e^{-2\sqrt{3}\Omega} \right]$$

- P□ , P□: canonical momenta conjugated to □, □
- Lapse function (gauge choice):

$$N = 24 e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega}$$

KS Cosmological Model – Classical and Quantum Models:

$c = \hbar = G = 1$

$$\{\Omega, P_\Omega\} = 1\ ,\ \{\beta, P_\beta\} = 1\ ,\ \{\Omega, \beta\} = \theta\ ,\ \{P_\Omega, P_\beta\} = \eta$$

θ~ LP2 ~ 1

□~LP-2~1

§ Equations of motion (Noncommutative):

$$\dot{\Omega} = -2P_\Omega \qquad \dot{P}_\Omega = 2\eta P_\beta - 96\sqrt{3}e^{-2\sqrt{3}\Omega}$$

$$\dot{\beta} = 2P_\beta - 96\sqrt{3}\theta e^{-2\sqrt{3}\Omega} \qquad \dot{P}_\beta = 2\eta P_\Omega$$

Constant of motion:

$$P_\beta + \eta\Omega = C$$

$$\left[\hat{\Omega}, \hat{\beta}\right] = i\theta\ ,\quad \left[\hat{P}_\Omega, \hat{P}_\beta\right] = i\eta\ ,\quad \left[\hat{\Omega}, \hat{P}_\Omega\right] = \left[\hat{\beta}, \hat{P}_\beta\right] = i$$

§ Non-unitary linear transformation, SW map:

$\xi \equiv \theta\eta < 1$

$$\hat{\Omega} = \lambda\hat{\Omega}_c - \frac{\theta}{2\lambda}\hat{P}_{\beta_c} \qquad \hat{\beta} = \lambda\hat{\beta}_c + \frac{\theta}{2\lambda}\hat{P}_{\Omega_c} \qquad \hat{P}_\Omega = \mu\hat{P}_{\Omega_c} + \frac{\eta}{2\mu}\hat{\beta}_c \qquad \hat{P}_\beta = \mu\hat{P}_{\beta_c} - \frac{\eta}{2\mu}\hat{\Omega}_c$$

§ ...tive ...on:

$$\left[-\left(-i\mu\frac{\partial}{\partial\Omega_c} + \frac{\eta}{2\mu}\beta_c\right)^2 + \left(-i\mu\frac{\partial}{\partial\beta_c} - \frac{\eta}{2\mu}\Omega_c\right)^2 - 48\exp\left[-2\sqrt{3}\left(\lambda\Omega_c + i\frac{\theta}{2\lambda}\frac{\partial}{\partial\beta_c}\right)\right]\right]\psi(\Omega_c, \beta_c) = 0$$

Solutions – Noncommutative WDW Equation:

From constraint: $\hat{A} = \frac{\hat{C}}{\sqrt{1-\xi}}$ ► $\mu \hat{P}_{\beta_c} + \frac{\eta}{2\mu}\hat{\Omega}_c = \hat{A}$

$$\left[\hat{P}_\beta + \eta\hat{\Omega}, \hat{H}\right] = \left[\hat{P}_\beta + \eta\hat{\Omega}, -\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48 e^{-2\sqrt{3}\hat{\Omega}}\right] = 0$$

§ Solutions of NCWDW Eq. are simultaneously eigenstates of Hamiltonian and constraint.

§ If □ a(□ c,□ c) is an eigenstate of operator Â with eigenvalue a□ IR:

$$\left(-i\mu\frac{\partial}{\partial\beta_c} + \frac{\eta}{2\mu}\Omega_c\right)\psi_a(\Omega_c,\beta_c) = a\psi_a(\Omega_c,\beta_c)$$ ► $$\psi_a(\Omega_c,\beta_c) = \Re(\Omega_c)\exp\left[\frac{i}{\mu}\left(a - \frac{\eta}{2\mu}\Omega_c\right)\beta_c\right]$$

§ Subtituting into NCWDW Eq. yields:

$$\mu^2\Re'' + \left(\eta\frac{\Omega_c}{\mu} - a\right)^2\Re - 48\exp\left[-2\sqrt{3}\frac{\Omega_c}{\mu} + \frac{\sqrt{3}\theta}{\lambda\mu}a\right]\Re = 0$$ + $$z = \frac{\Omega_c}{\mu} \rightarrow \frac{d}{dz} = \mu\frac{d}{d\Omega_c}$$ $$\phi(z) \equiv \Re(\Omega_c(z))$$

$$\phi''(z) + (\eta z - a)^2\phi(z) - 48\exp\left[-2\sqrt{3}z + \frac{\sqrt{3}\theta}{\lambda\mu}a\right]\phi(z) = 0$$

Model - Potential:

$$\hbar = c = k = G = 1$$

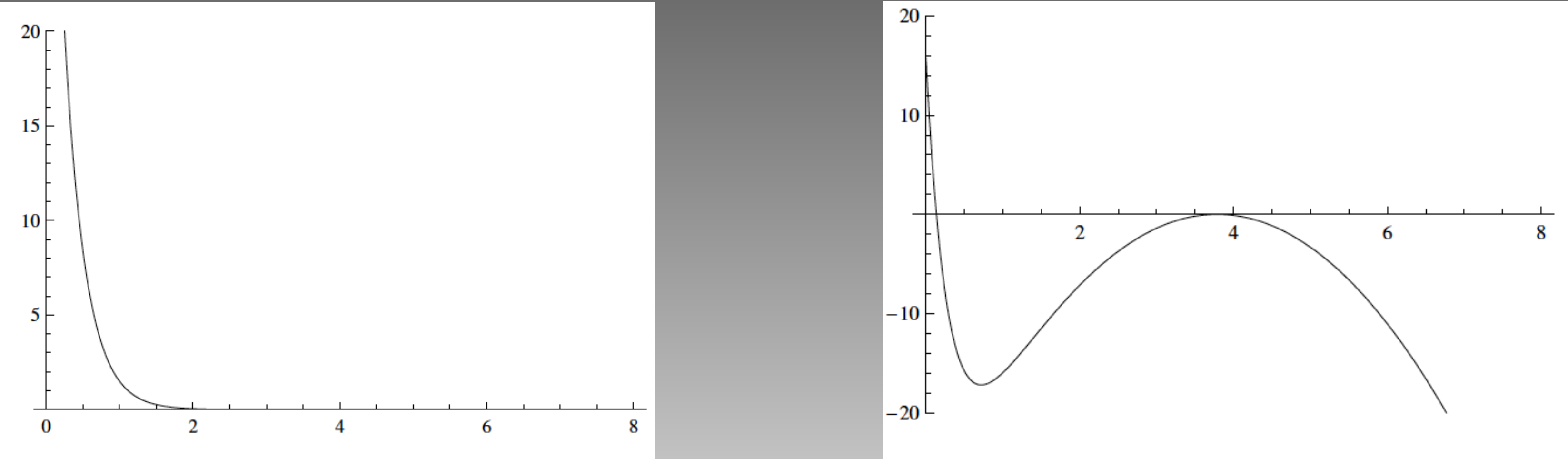
$$V(z) = 48\exp\left[-2\sqrt{3}z + \frac{\sqrt{3}\theta}{\lambda\mu}a\right] - (\eta z - a)^2$$

$$x = z - \frac{\theta}{2\lambda\mu}a$$

§ Potential function:

$$V(x) = 48\exp\left(-2\sqrt{3}x\right) - (\eta x - c)^2$$

$$c = P_\beta(0) + \eta\Omega(0)$$



(a) $\eta = 0$ and $c = 0.01$

(b) $\eta = 1.5$ and $c = 5.68$

For η values fairly typical and non-zero, potential has a local minimum and maximum.

Model - Potential:

$$\hbar = c = k = G = 1$$

- Local minimum:

$$\frac{dV}{dx}|_{x_0} = -96\sqrt{3}\exp\left(2\sqrt{3}x_0\right) - 2\eta^2 x_0 + 2\eta c = 0$$

- Solution (implicit):

$$\exp\left(-2\sqrt{3}x_0\right) = \zeta D - \frac{\zeta^2}{\sqrt{3}}x_0$$

$$\zeta = \eta/4\sqrt{3}$$

$$D = c/12$$

$$6\exp\left(-2\sqrt{3}x_0\right) - \zeta^2 > 0 \Leftrightarrow x_0 < -\frac{1}{\sqrt{3}}\ln\left(\frac{\zeta}{\sqrt{6}}\right)$$

- Potential function in second order in x-x0:

$$V(x) = 48(6e^{-2\sqrt{3}x_0} - \zeta^2)(x - x_0)^2 + 48e^{-2\sqrt{3}x_0} - (\eta x_0 - c)^2$$

- NCWDW Equation:

$$-\frac{1}{2}\frac{d^2\phi}{dx^2} + 24(6e^{-2\sqrt{3}x_0} - \zeta^2)(x - x_0)^2\phi + [24e^{-2\sqrt{3}x_0} - \frac{1}{2}(\eta x_0 - c)^2]\phi = 0$$

Model - Potential:

$$\hbar = c = k = G = 1$$

§ Comparing with Schrodinger equation of harmonic oscillator:

$$V_{NC}(y) = 24(6e^{-2\sqrt{3}x_0} - \zeta^2)y^2$$

$$y = x - x_0$$

§ Quantum correction to potential:

$$\frac{\beta_{BH}}{24} V''_{NC}(y) = 2\beta_{BH}(6e^{-2\sqrt{3}x_0} - \zeta^2)$$

§ Potential function:

$$U_{NC}(y) = 24(6e^{-2\sqrt{3}x_0} - \zeta^2)\left(y^2 + \frac{\beta_{BH}}{12}\right)$$

§ Partition Function:

$$Z_{NC} = \sqrt{\frac{1}{48(6e^{-2\sqrt{3}x_0} - \zeta^2)}} \frac{1}{\beta_{BH}} \exp\left[-2\beta_{BH}^2 \left(6e^{-2\sqrt{3}x_0} - \zeta^2\right)\right]$$

Thermodynamical properties:

$\hbar = c = k = G = 1$

- Noncommutative internal energy :

$$\bar{E}_{NC} = \frac{1}{\beta_{BH}} + 4(6e^{-2\sqrt{3}x_0} - \zeta^2)\beta_{BH}$$

- Noncommutative Temperature (E=M), M>>1:

$$T_{BH} = \frac{4}{M}(6e^{-2\sqrt{3}x_0} - \zeta^2)$$

$$x_0 = 1.8478 \qquad \eta = 0.025$$

$$T_{BH} = \frac{1}{8\pi M}$$

- Noncommutative Entropy (neglecting terms proportional to η2/M2):

c=12D=5.
68

$$S_{BH} \simeq \frac{M^2}{2b(\zeta)} + \ln\frac{\sqrt{b(\zeta)}}{M\sqrt{3}}$$

Singularity, t=r=0:

$\hbar = c = k = G = 1$

- By the identification between metrics:

$$t = 0,\ \Omega \to +\infty \text{ and } \beta \to +\infty$$

- Study the limit: $\lim_{\Omega_c,\beta_c \to +\infty} \psi(\Omega_c, \beta_c)$ $\qquad \psi(\Omega_c,\beta_c) = \int da\, C(a)\psi_a(\Omega_c,\beta_c)$

- NCWDW equation in this limit:

$$\phi_a''(z) + (\eta z - a)^2 \phi_a(z) = 0$$

$$\left\{ -\frac{\partial^2}{\partial z^2} - (\eta z - a)^2 \right\} \phi_a(z) = 0 \quad \Longleftrightarrow \quad \left\{ -\frac{\partial^2}{\partial \tilde{z}^2} - \eta^2 \tilde{z}^2 \right\} \tilde{\phi}_a(\tilde{z}) = 0$$

$$\tilde{z} = z - \tfrac{a}{\eta} \text{ and } \tilde{\phi}_a(x) = \phi_a(x + \tfrac{a}{\eta})$$

Inverted harmonic oscillator: self-adjoint Hamiltonian with a continuous spectrum.

Model – Feynman-Hibbs procedure:

$$\hbar = c = k = G = 1$$

- § Solution to NCWDW equation in t=r=0:

$$\tilde{\phi}_a(\tilde{z}) \sim \frac{1}{\tilde{z}^{1/2}} \exp\left[\pm i \frac{\eta}{2} \tilde{z}^2\right]$$

- § For all a:

$$\lim_{z \to +\infty} \phi_a(z) = \lim_{z \to +\infty} \tilde{\phi}_a\left(z - \frac{a}{\eta}\right) = 0 \quad \Longrightarrow \quad \lim_{\Omega_c, \beta_c \to +\infty} \psi_a(\Omega_c, \beta_c) = 0$$

$$\lim_{\Omega_c, \beta_c \to +\infty} \psi(\Omega_c, \beta_c) = 0$$

Necessary condition to provide a quantum regularization of the classical singularity

Model – Feynman-Hibbs procedure:

$\hbar = c = k = G = 1$

Is the probability of finding the BH at the singularity zero?

- Wave function oscillatory for βC: Fix βC-hypersurface.
  - Probability of finding the BH at the singularity:

$$P(r=0,t=0) = \lim_{\Omega_c,\beta_c \to +\infty} \int^{+\infty} \ldots |^2 d\Omega_c' \simeq \lim_{\Omega_c \to +\infty} \int_{\Omega_c}^{+\infty} |\phi_a(\frac{\Omega_c'}{\mu})|^2 d\Omega_c'$$

DIVERGES!

Inverted harmonic oscillator displays non-normalizable eigenstates!

Noncommuativity of this form cannot be regarded as the final answer for the singularity problem of the Schwarzschild BH!

Singularity, t=r=0:

$$\hbar = c = k = G = 1$$

§ Phase-Space Noncanonical Noncommutativity:

$$\left[\hat{\Omega}, \hat{\beta}\right] = i\theta\left(1 + \epsilon\theta\hat{\Omega} + \frac{\epsilon\theta^2}{1+\sqrt{1-\xi}}\hat{P}_\beta\right)$$

$$\left[\hat{P}_\Omega, \hat{P}_\beta\right] = i\left(\eta + \epsilon(1+\sqrt{1-\xi})^2\hat{\Omega} + \epsilon\theta(1+\sqrt{1-\xi})\hat{P}_\beta\right)$$

$$\left[\hat{\Omega}, \hat{P}_\Omega\right] = \left[\hat{\beta}, \hat{P}_\beta\right] = i\left(1 + \epsilon\theta(1+\sqrt{1-\xi})\hat{\Omega} + \epsilon\theta^2\hat{P}_\beta\right),$$

§ F

$$E = -\frac{\theta}{1+\sqrt{1-\xi}}F\ ,\ F = -\frac{\lambda}{\mu}\epsilon\sqrt{1-\xi}\left(1+\sqrt{1-\xi}\right)$$

$$V(z) \sim -F^2\mu^4 z^4,$$

CWDW Equation:

§ Square integrable

§ Probability vanishes!

$$\phi_a(z) \sim \frac{1}{z}\exp\left[\pm i\frac{F\mu^2}{3}z^3\right]$$

Solutions of the new NCWDW equation would display zero probability at the singularity .

Conclusions:

- Kantowski-Sachs used to study interior of a Schwarzschild BH ($r<2M$)

  - Thermodynamical quantities and singularity analyzed

  - Momentum noncommutativity seems crucial:
  - Potential with quadratic term allowing Feynman-Hibbs procedure
  - Noncommutative Temperature and Noncommutative Entropy

  - Singularity $t=r=0$:
  - Inverted harmonic oscillator
  - Wave function vanishes but is not square integrable with phase space canonical NC.