Second-order symmetric spacetimes in arbitrary dimension

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   - Riemannian vs. Lorentzian case.

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RIEMANNIAN CASE

- Locally symmetric spaces: $\nabla_\rho R^\alpha{}_{\beta\lambda\mu} = 0$ -Cartan 1926,1927-

$$\nabla_{\rho_1} \dots \nabla_{\rho_n} R^\alpha{}_{\beta\lambda\mu} = 0 \Longrightarrow \nabla_\rho R^\alpha{}_{\beta\lambda\mu} = 0$$ -Tanno 1972-

(*Essential tool:* orthogonal de Rham decomposition.)

- **Natural generalization of locally symmetric spaces:**

  Semi-symmetric spaces: $\nabla_{[\rho}\nabla_{\sigma]} R^\alpha{}_{\beta\lambda\mu} = 0$. -Szabó 1982,1985-

- **Hierarchy of conditions:**

$R^\alpha{}_{\beta\lambda\mu} = K(g_{\lambda\beta}\delta^\alpha_\mu - g_{\mu\beta}\delta^\alpha_\lambda)$	$\nabla_\rho R^\alpha{}_{\beta\lambda\mu} = 0$	$\nabla_{[\rho}\nabla_{\sigma]} R^\alpha{}_{\beta\lambda\mu} = 0$
Constant curvature	Symmetric	Semi-symmetric

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- **IN 4-DIMENSIONS:**

THEOREM *[OFB, Sánchez M, Senovilla JMM 2009]*

A 2-symmetric non-symmetric **4-dimensional** spacetime (M, g) is locally isometric to $\mathbb{R}^4$ endowed with the metric

$$ds^2 = -2du(dv + Hdu) + dx^2 + dy^2$$

where $H(u, x, y) = (\alpha_1 u + \beta_1)x^2 + (\alpha_2 u + \beta_2)y^2 + (\alpha_3 u + \beta_3)xy$ for some constants $\{\alpha_A, \beta_A\}_{A=1,2,3}$ with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0$.

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- **IN ARBITRARY DIMENSIONS**:

THEOREM *[OFB, Sánchez M, Senovilla JMM 2010]*

A 2-symmetric non-symmetric **n-dimensional** spacetime (M, g) is locally isometric to $\mathbb{R}^n$ endowed with the metric

$$ds^2 = -2du(dv + Hdu) + \sum_{i=2}^{d}(dx^i)^2 + \sum_{i,j=d+1}^{n-1} g_{ij}(x^{d+1}, \dots, x^{n-1})dx^i dx^j$$

where $H(u, x, y) = \sum_{i,j=2}^{d}(\alpha_{ij} u + \beta_{ij})x^i x^j$ for some constants $\{\alpha_{ij}, \beta_{ij}\}_{i,j=1,\dots,d}$, and $\sum_{i,j=d+1}^{n-1} g_{ij}(x^{d+1}, \dots, x^{n-1})dx^i dx^j$ (non-flat) locally symmetric.

2-SYMMETRIC SPACETIMES: STEPS FOR THE SEARCH

- 1st step: when $\nabla_\rho \nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0 \Longrightarrow \nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0$???

THEOREM *[Senovilla J M M, 2008]*

Let $D \in M$ be a simply connected domain of an n-dimensional 2-symmetric Lorentzian manifold (M, g). Then, if there is **no null covariantly constant vector field** on D, (D, g) is in fact locally symmetric (i.e., $\nabla_\rho R^\alpha{}_{\beta\lambda\mu} = 0$).

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The spacetimes with a **null covariantly constant vector field** have the metric:

$$ds^2 = -2du(dv + H(u, x^k)du + W_i(u, x^k)dx^i) + g_{ij}(u, x^k)dx^i dx^j, \quad i, j \in \{2, \dots, n-1\} \quad (1)$$

Conclusion: the set of *2-symmetric non-symmetric* spacetimes is contained in the set of the spacetimes with metric (1).

- 2^{nd} Step: make a suitable choice of moving frame in order to write the equations of 2-symmetry.

The obtained first result:

For a 2-symmetric spacetime, g_{ij} is locally symmetric on each slice $u = u_0$

- 3rd Step: solve the equations:
  - In four dimensions: the study is reduced to **an analysis of the Petrov types** of the spacetime, and it is almost straighforward to prove that $g = dx^2 + dy^2$.
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  - In arbitrary dimension: The equations are much more involved and a **deeper analysis is needed.**
    - After lengthly computations $\longrightarrow$partial decomposition of the equations
    - Thanks to such decomposition, we find:

    1. the equations of Local Symmetry ($\nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0$)
       -already classified by Cahen&Wallach-
    2. the equations of 2-symmetry for a plane wave
       -easily solvable-

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