# The Einstein field equations for cylindrically symmetric elastic configurations

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Dedicated to the memory of Brian Edgar

# Relativistic elasticity

General relativistic elasticity was formulated in the mid-20th century due to the necessity to study astrophysical problems such as deformations of neutron star crusts, which can be modelled by axially symmetric metrics.

Relevant contributions to the theory of general relativistic elasticity were given, for example, by Carter and Quintana (1972), Kijowski and Magli (1992), Beig and Schmidt (2003), Karlovini and Samuelsson (2003).

The here presented work is based on Magli (1993) and Brito, Carot and Vaz (2010).

# Relativistic elasticity

## Configuration mapping

The space-time configuration of the material is described by the mapping

$$\Psi : M \longrightarrow X.$$

- $(M, g_{ab})$ space-time with coordinate system $\{x^a\}$, $a = 0, 1, 2, 3$
- $(X, \gamma_{AB})$ material space with material metric $\gamma_{AB}$ and coordinate system $\{y^A\}$, $A = 1, 2, 3$

# Relativistic elasticity

## Pulled-back material metric

$$k_{ab} = \Psi^* \gamma_{AB} = y^A_a y^B_b \gamma_{AB}$$

$y^A_a = \dfrac{\partial y^A}{\partial x^a}$ is the relativistic deformation gradient.

## Velocity field of the matter

The velocity field of the matter $u^a \in T_p M$ is defined by the conditions

$$u^a y^A_a = 0,\ u^a u_a = -1,\ u^0 > 0.$$

# Relativistic elasticity

## Relativistic strain tensor

The operator $K^a{}_b = -u^a u_b + k^a{}_b$ can be used to measure the state of strain of the material.

The relativistic strain tensor is defined by

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab}) = \frac{1}{2}(g_{ab} - K_{ab}),$$

where $h_{ab} = g_{ab} + u_a u_b$.

The material is in an unstrained state if $s_{ab} = 0$.

# Relativistic elasticity

## Energy-momentum tensor

$$T^a{}_b = \rho\, \delta^a_b - \frac{\partial \rho}{\partial I_3} \det K\, h^a{}_b + \left( \mathrm{Tr} K\, \frac{\partial \rho}{\partial I_2} - \frac{\partial \rho}{\partial I_1} \right) k^a{}_b - \frac{\partial \rho}{\partial I_2}\, k^a{}_c\, k^c{}_b$$

- $\rho = \epsilon v$ energy density
- $\epsilon$ particle number density
- $v = v(I_1, I_2, I_3)$ constitutive equation

$I_1$, $I_2$ and $I_3$ are the invariants of $K$:

$I_1 = \frac{1}{2}\left(\mathrm{Tr}K - 4\right)$, $I_2 = \frac{1}{4}\left[\mathrm{Tr}K^2 - \left(\mathrm{Tr}K\right)^2\right] + 3$, $I_3 = \frac{1}{2}\left(\det K - 1\right)$,

which can be written in terms of the eigenvalues of $K$.

1. Relativistic Elasticity
2. Cylindrically symmetric elastic configurations
3. Einstein field equations

## Cylindrically symmetric space-time

### Space-time $(M, g)$

- Cylindrically symmetric metric $g$

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + e^{2\mu(r)}dz^2 + e^{2\psi(r)}d\phi^2$$

- Coordinates: $x^a = \{t, r, z, \phi\}$

- Pulled-back material metric $k$

$$d\Sigma^2 = d\tilde{r}^2 + dz^2 + \tilde{r}^2 d\phi^2$$

- Coordinates on $X$: $y^A = \{\tilde{r}, \tilde{z}, \tilde{\phi}\}$,
  $\tilde{r} = \tilde{r}(r) = r, \ \tilde{z} = z, \ \tilde{\phi} = \phi$

## Cylindrically symmetric space-time

The operator $K^a{}_b = -u^a u_b + k^a{}_b$ is given by

$$K^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\mu} & 0 & 0 \\ 0 & 0 & e^{-2\mu} & 0 \\ 0 & 0 & 0 & r^2 e^{-2\psi} \end{pmatrix}.$$

It has one eigenvalue equal to 1 and the other eigenvalues are

$$\eta = e^{-2\mu}$$
$$\tau = r^2 e^{-2\psi},$$

where $\eta$ has algebraic multiplicity two.

# Cylindrically symmetric space-time

## Invariants of $K$

$$I_1 = \frac{1}{2}\left(\mathrm{Tr}K - 4\right) = \frac{1}{2}\left(2\eta + \tau - 3\right)$$
$$I_2 = \frac{1}{4}\left[\mathrm{Tr}K^2 - \left(\mathrm{Tr}K\right)^2\right] + 3 = -\frac{1}{2}\left(\eta^2 + 2\eta\tau + 2\eta + \tau\right) + 3$$
$$I_3 = \frac{1}{2}\left(\det K - 1\right) = \frac{1}{2}\left(\eta^2\tau - 1\right)$$

# Cylindrically symmetric space-time

## Energy-momentum tensor

$$T^0{}_0 = \rho,$$

$$T^1{}_1 = \rho - \frac{\partial\rho}{\partial I_3}\,\eta^2\tau + \frac{\partial\rho}{\partial I_2}(1+\eta+\tau)\eta - \frac{\partial\rho}{\partial I_1}\eta,$$

$$T^2{}_2 = T^1{}_1,$$

$$T^3{}_3 = \rho - \frac{\partial\rho}{\partial I_3}\,\eta^2\tau + \frac{\partial\rho}{\partial I_2}(1+2\eta)\tau - \frac{\partial\rho}{\partial I_1}\tau.$$

# Einstein-field equations

$$G^0{}_0 = 8\pi T^0{}_0: \qquad \frac{1}{8\pi}\,\frac{\mu'' + \psi'' + \psi'^2}{e^{2\mu}} = \epsilon v$$

$$G^1{}_1 = 8\pi T^1{}_1: \qquad \frac{1}{8\pi}\,\frac{\mu'\nu' + \mu'\psi' + \nu'\psi'}{e^{2\mu}} = -\epsilon\eta\,\frac{\partial v}{\partial \eta}$$

$$G^2{}_2 = 8\pi T^2{}_2: \qquad \frac{1}{8\pi}\,\frac{\nu'^2 + \nu'' + \psi'' + \psi'^2 + \nu'\psi' - \mu'\nu' - \mu'\psi'}{e^{2\mu}} = -\epsilon\eta\,\frac{\partial v}{\partial \eta}$$

$$G^3{}_3 = 8\pi T^3{}_3: \qquad \frac{1}{8\pi}\,\frac{\nu'^2 + \nu'' + \mu''}{e^{2\mu}} = -2\epsilon\tau\,\frac{\partial v}{\partial \tau}$$

# Einstein-field equations

Setting $E = \ln \eta = -2\mu$ and $T = \ln \tau = 2 \ln r - 2\psi$, one gets

$$\frac{\partial \ln v}{\partial E} = -\frac{\mu'\nu' + \mu'\psi' + \nu'\psi'}{\mu'' + \psi'' + \psi'^2}$$

$$\frac{\partial \ln v}{\partial T} = -\frac{1}{2}\,\frac{\nu'^2 + \nu'' + \mu''}{\mu'' + \psi'' + \psi'^2}.$$

Since $T^2{}_2 = T^1{}_1$, it follows that

$$2\mu'\nu' + 2\mu'\psi' - \nu'^2 - \nu'' - \psi'' - \psi'^2 = 0.$$

## Einstein-field equations

In order for a constitutive equation $v = v(\eta, \tau) = v(E, T)$ to exist, it must be that

$$\frac{\partial^2 \ln v}{\partial T \partial E} = \frac{\partial^2 \ln v}{\partial E \partial T}.$$

Therefrom, one obtains

$$\frac{\partial}{\partial T}\left[-\frac{1}{\mu'}\left(\frac{1}{r} - \psi'\right)\right]\frac{\partial \ln v}{\partial T} = 0.$$

## Einstein-field equations

The case:

$$\frac{\partial}{\partial T}\left[-\frac{1}{\mu'}\left(\frac{1}{r}-\psi'\right)\right]=0 \quad \text{and} \quad \frac{\partial \ln v}{\partial T}=0$$

leads to the conditions

$$\psi=\ln(r)+k_0\mu+k_1 \quad \text{and} \quad T^3_3=0.$$

# Einstein-field equations

In order to avoid singularities at the axis of symmetry, one must have $e^{2\psi} = r^2 L(r)$, where $L(r) \neq 0$ for $r = 0$, Carot (2000).

Then, one has

$$\psi(r) = \ln(r) + \frac{1}{2}\ln(L)$$
$$\mu(r) = \frac{1}{2}\ln(L)$$
$$\nu(r) = -\frac{1}{4}\ln(L) \ + \ \text{constant.}$$

# Einstein-field equations

The function $L(r)$ must satisfy the condition

$$6L'^2Lr - 8L'''L^2r^2 - 8L''L^2r + 16L''L'Lr^2 - 9L'^3r^2 + 8L'L^2 = 0.$$

The constitutive function takes the form

$$v(r) = c\ \exp\left(\int \frac{L'^2}{-3L'^2r + 4LL' + 4L''Lr}dr\right).$$

## References


- Beig, R. and Schmidt, B., *Class. Quantum Grav.*, 20, 889-904, (2003).
- Brito, I., Carot, J., Vaz, E.G.L.R., *Gen. Rel. Grav.*, DOI - 10.1007/s10714-010-0980-x, http://www.springerlink.com/content/33t2056301904175, (2010).
- Carot, J., *Class. Quantum Grav.*, 17, 2675-2690, (2000).
- Carter,B. and Quintana, H., *Proc. R. Soc. Lond. A*, 331, 57-83, (1972).
- Karlovini, M. and Samuelsson, L., *Class. Quantum Grav.*, 20, 3613-3648, (2003).
- Kijowski, J. and Magli, G., *J. Geom. Phys.*, 9, 207-223, (1992).
- Magli, G., *Gen. Rel. Grav.*, 25, 441-460, (1993).
- Magli, G., *Gen. Rel. Grav.*, 25, 1277-1293, (1993).