Unruh Effect and the Breakdown of the Conformal Symmetry

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Outline

Vacuum radiation as a consequence of space-time mutilation

The existence of event horizons in passing to accelerated frames of reference leads to unitarily inequivalent representations of the quantum field canonical commutation relations and to an ill-definition of particles depending on the state of motion of the observer.

Field decompositions and vacua

2D-Massless free Klein-Gordon field

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) = 0\,,$$

Field decomposition into positive and negative frequency modes:

$$\hat{\phi}(x) = \int dk(\hat{a}_k f_k(x) + \hat{a}_k^\dagger f_k^*(x))$$

Minkowski vacuum: $\hat{a}_k|0\rangle_M = 0\,\forall k$.

Rindler coordinate transformations

(Global mutilation of the space-time):

$$t = a^{-1} e^{az'} \sinh(at'), \quad z = a^{-1} e^{az'} \cosh(at'),$$

the worldline $z' = 0$ has constant acceleration a.
Field decomposition into Rindler positive and negative frequency modes:

$$\hat{\phi}(x') = \int dq(\hat{a}'_q f'_q(x') + \hat{a}'^{\dagger}_q f'^{*}_q(x'))$$

Rindler vacuum: $\hat{a}'_q |0\rangle_R = 0 \, \forall q$.

Bogolyubov transformation

$$\hat{a}'_q = \int dk \left(\alpha_{qk} \hat{a}_k + \beta_{qk} \hat{a}_k^\dagger \right).$$

$$\alpha_{qk} = \langle f'_q | f_k \rangle, \quad \beta_{qk} = \langle f'_q | f_k^* \rangle$$

Average number of Rindler particles in the Minkowski vacuum

$$N_R = \langle 0 | \hat{N}_R | 0 \rangle_M = \langle 0 | \int dq \hat{a}'^\dagger_q \hat{a}'_q | 0 \rangle_M = \int dk dq |\beta_{qk}|^2$$

In the second quantized theory, the vacuum states $|0\rangle_M$ and $|0\rangle_R$ are not identical if the coefficients $\beta_{qk} \neq 0$.

Actually, uniformly accelerated observers in Minkowski spacetime (Rindler observers) associate a thermal bath of Rindler particles to the no-particle state of inertial observers (Minkowski vacuum $|0\rangle_M$) with temperature

$$T = \frac{\hbar a}{2\pi c k_B}$$

Outline

Vacuum radiation as a spontaneous breakdown of the conformal symmetry

Poincaré invariant pseudo-vacua (coherent states of conformal zero modes) are not invariant under special conformal transformations (relativistic uniform accelerations) and radiate as a black body.

The conformal group $SO(4,2)$ is comprised of Poincaré (spacetime translations $b^\mu \in \mathbb{R}^{1,3}$ and Lorentz $\Lambda^\mu_\nu \in SO(3,1)$) transformations augmented by dilations ($\rho = e^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, SCT, $c^\mu \in \mathbb{R}^{1,3}$) which, in Minkowski spacetime, have the following realization:

$$\begin{array}{ll} x'^\mu = x^\mu + b^\mu, & x'^\mu = \Lambda^\mu_\nu(\omega) x^\nu, \\ x'^\mu = \rho x^\mu, & x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2 x^2}, \end{array}$$

respectively.

The interpretation of special conformal transformations

$$x'^{\mu} = \frac{x^{\mu} + c^{\mu} x^2}{1 + 2cx + c^2 x^2}$$

as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by: Hill 1945, Fulton-Rohrlich-Witten 1962, Boya-Cerveró 1975, etc., although alternative meanings have also been proposed (Weyl 1922, Kastrup 1966).

For $c^\mu = (0,0,0,\alpha)$, and the temporal path $x^\mu = (t,0,0,0)$, the STC reads:

$$t' = \frac{t}{1-\alpha^2 t^2}\,, \qquad z' = \frac{\alpha t^2}{1-\alpha^2 t^2}.$$

Writing z' in terms of t' gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

$$z' = \frac{1}{a}(\sqrt{1+a^2 t'^2} - 1)$$

with $a = 2\alpha$

Outline

Four cover of $SO(4,2)$

$$SU(2,2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}) : g^\dagger \gamma g = \gamma, \det(g) = 1 \right\}$$

γ hermitian form of signature $(++,--)$. The group $SU(2,2)$ acts transitively on the compactified Minkowski space $\mathbb{M}_4 = U(2)$, with (matrix) coordinates Z, as

$$Z \to Z' = (AZ + B)(CZ + D)^{-1}.$$

Setting $Z = z_\mu \sigma^\mu$ (σ^μ Pauli matrices), STC correspond to $A = D = I, B = 0, C = c_\mu \sigma^\mu$:

$$Z' = Z(CZ + I)^{-1} \leftrightarrow z'^\mu = \frac{z^\mu + c^\mu z^2}{1 + 2cz + c^2 z^2}$$

Unirreps of the Conformal Group: Discrete Series

We shall consider the complex extension of $\mathbb{M}_4 = U(2)$ to the 8-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\}$$

and the Unirrep

$$[U_\lambda(g)\phi](Z) = |CZ + D|^{-\lambda}\phi(Z') \qquad (1)$$

on the space $\mathcal{H}_\lambda(\mathbb{D}_4)$ of square-integrable holomorphic functions ϕ with invariant integration measure

$$d\mu_\lambda(Z, Z^\dagger) = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)\det(I - ZZ^\dagger)^{\lambda-4}|dZ|,$$

where the label $\lambda \geq 4$ is the conformal, scale or mass dimension.

The Hilbert Space of the Conformal Particle

The infinite set of homogeneous polynomials

$$\varphi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\,\det(Z)^m \mathcal{D}^j_{q_1,q_2}(Z),$$

with $\mathcal{D}^j_{q_1,q_2}(Z)$ the standard Wigner's $\mathcal{D}$-matrices ($j \in \mathbb{N}/2$), verifies the following closure relation (the reproducing Bergman kernel):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} \overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I - Z^\dagger Z')^\lambda}$$

and constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D}_4)$.

Hamiltonian and Energy Spectrum

The Hamiltonian operator (dilation generator) is:

$$H = \lambda + \sum_{i,j=1}^{2} Z_{ij} \frac{\partial}{\partial Z_{ij}} = \lambda + z_\mu \frac{\partial}{\partial z_\mu}.$$

The energy spectrum is:

$$H\varphi_{q_1,q_2}^{j,m} = E_n^\lambda \varphi_{q_1,q_2}^{j,m}, \quad E_n^\lambda = \lambda + n, \quad n = 2j + 2m.$$

Each energy level E_n^λ is $(n+1)(n+2)(n+3)/6$ times degenerated. The spectrum is equi-spaced and bounded from below, with ground state $\varphi_{0,0}^{0,0} = 1$ and zero-point energy $E_0^\lambda = \lambda$
.

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The Ground State is Poincaré-Stable and Polarized by Accelerations

Excited (accelerated) ground state:

$$\tilde{\varphi}^{0,0}_{0,0}(Z) = [U_\lambda(g)\varphi^{0,0}_{0,0}](Z) = \det(CZ+D)^{-\lambda} \left(= \varphi^{0,0}_{0,0}(Z) \text{ if } C = 0\right)$$

Using the Bergman kernel expansion, we can decompose the accelerated ground state as ($D = I$):

$$\tilde{\varphi}^{0,0}_{0,0}(Z) = \sum_{j\in\mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1,q_2=-j}^{j} \varphi^{j,m}_{q_2,q_1}(-C)\varphi^{j,m}_{q_1,q_2}(Z)$$

with $\varphi^{j,m}_{q_2,q_1}(-C)$ the probability amplitude of finding the accelerated ground state in the excited level $\varphi^{j,m}_{q_1,q_2}$ of energy $E^\lambda_n = \lambda + 2j + 2m = \lambda + n$

Mean Energy of the accelerated ground state:

$$\begin{aligned}\mathcal{E}(C) &= \frac{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}|\varphi^{j,n}_{q_1,q_2}(C)|^2(\lambda+2j+2m)}{\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}|\varphi^{j,n}_{q_1,q_2}(C)|^2} \\ &= \lambda\frac{1-\det(C^\dagger C)}{\det(I-C^\dagger C)} = \lambda + 2\lambda\frac{\alpha^2}{1-\alpha^2} = \mathcal{E}(\alpha), \quad \text{for} \quad C=\alpha\sigma^3\end{aligned}$$

Let us show how we can see our accelerated ground state as a thermal state and compute the entropy and temperature as a function of the acceleration α, so that the mean energy $\mathcal{E}(\alpha)$ acquires the (Planckian) form of an Einstein solid.

Partition Function and Entropy:

The partition function

$$Z(\alpha)=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}|\varphi^{j,m}_{q_1,q_2}(\alpha)|^2=\sum_{n=0}^{\infty}\underbrace{\binom{n+2\lambda-1}{2\lambda-1}\alpha^{2n}}_{p_n(\alpha)Z(\alpha)}=(1-\alpha^2)^{-2\lambda}.$$

matches that of the Einstein solid with 2λ degrees of freedom for

$$\alpha^2(T)\equiv e^{-\frac{h\nu}{k_BT}}, \tag{2}$$

The entropy of our accelerated ground state will be:

$$S(\alpha)=-k_B\sum_{n=0}^{\infty}p_n(\alpha)\ln p_n(\alpha)=-k_B\lambda\left(\frac{4\alpha^2\ln(\alpha)}{1-\alpha^2}+2\ln(1-\alpha^2)\right). \tag{3}$$

Temperature and "Maximal Acceleration"

The temperature is given by the thermodynamic relation:

$$T = k_B T_E \frac{d\mathcal{E}(\alpha)}{dS(\alpha)} = -\frac{T_E}{2\ln(\alpha)} \Rightarrow \alpha^2 = e^{-T_E/T} \equiv \frac{a^2}{a_{\max}^2}, \qquad (4)$$

where we have introduced the Einstein temperature $T_E = h\nu/k_B$ and the maximal acceleration $a_{\max} = c^2/\ell$ in order to give dimensions.

Second-Quantized Theory and Conformal Zero Modes

The Fourier coefficients a_n (and $\bar{a}_n$) of the expansion in energy modes of a state

$$\phi = \sum_n a_n \varphi_n,$$

are promoted to annihilation $\hat{a}_n$ (and creation $\hat{a}_n^\dagger$) operators. The fact that the ground state of first quantization, φ_0, is invariant under Poincaré transformations imposes $\hat{a}_0$ to behave as a multiple of the identity in the broken theory, which means that Poincaré θ-vacua

$$\hat{a}_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = e^{\theta\hat{a}_0 - \bar{\theta}\hat{a}_0^\dagger}|0\rangle$$

are coherent states of conformal zero modes.

Accelerated Poincaré θ-Vacuum as a Thermal Bath

The average number of particles with energy E_n in the accelerated vacuum

$$|\theta'\rangle = e^{\theta \hat{a}'_0 - \bar{\theta}\hat{a}'^{\dagger}_0}|0\rangle$$

with

$$\hat{a}'_0 = \sum_{n=0}^{\infty} \varphi_n(\alpha)\hat{a}_n$$

is:

$$N(\alpha) = \langle\theta'|\hat{a}^{\dagger}_n\hat{a}_n|\theta'\rangle = |\theta|^2|\varphi_n(\alpha)|^2$$

and the mean energy per mode reproduces that of the Black-Body spectrum for $\alpha^2 = e^{-T_E/T}$ and the Einstein temperature $T_E = h\nu/k_B$

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Thank you for your attention!