Integrating geodesic flows : finding supergravity cosmologies and black holes

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Based on: W.C, P. Fré, Alexender Sorin, **arXiv:0904.0801**
W. C, M. Trigiante, T. Van Riet, Jan.R., **arXiv:0903.2777**
W. C, P. Fré, A. Sorin, M. Trigiante, T. Van Riet, Jan.Rosseel, **arXiv:1007.3209**
W.C, M. Walton, **to appear**

ERE2010, 06/09/2010

Outline

Introduction : Goals

- Main goal : finding p-brane type solutions of supergravity theories in an algorithmic manner. We consider both time-like branes, as well as space-like branes.
- Strategy : use the fact that, in case symmetry is present, the branes are described by geodesic motion on a certain moduli space (after performing a certain dimensional reduction).
- In case the moduli space is a *symmetric space*, the geodesic equations that describe *both* time-like and space-like branes can be written in a specific form : the Lax pair form.
- This rewriting establishes integrability. The explicit integration can moreover be done in an algorithmic manner.
- In this way, one can find (after oxidation) cosmological solutions of SUGRA, as well as e.g. black hole solutions (both BPS and non-BPS) without relying on supersymmetry arguments.

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Branes as geodesics on moduli space

- p-brane solutions in d dimensions are charged electrically under A_{p+1} or magnetically under A_{d-p-3}.

$$
\begin{aligned}
\mathrm{d}s_d^2 &= \mathrm{e}^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{e}^{2B(r)}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-p-2}^2) \quad (\mathrm{time-like})\,,\\
\mathrm{d}s_d^2 &= \mathrm{e}^{2A(t)}\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{e}^{2B(t)}(-\mathrm{d}t^2 + r^2\mathrm{d}\Sigma_{d-p-2}^2) \quad (\mathrm{space-like})
\end{aligned}
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- Transversal symmetries : $\mathrm{SO}(d-p-1)$ or $\mathrm{SO}(d-p-2,1)$
- Worldvolume symmetries contain an $\mathbb{R}^{p+1}$ subgroup of translations $\longrightarrow$ matter fields are translation invariant.
- For the purpose of finding solutions : effectively dimensionally reduce the solutions over its worldvolume :

$$p\text{-brane in } d \text{ dim.} \longrightarrow -1\text{-brane in } D = d-p-1 \text{ dim.}$$

$$S = \int \mathrm{d}^D x\,\sqrt{|g|}\left\{R - \frac{1}{2}G_{ij}(\phi)\partial\phi^i\partial\phi^j\right\}\,.$$

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- E.o.m.'s for the scalars decouple from these for the metric. The e.o.m.'s for the scalars reduce to *geodesic equations* in the moduli space with metric $G_{ij}(\phi)$.
- Main difference between time-like and space-like branes:

time-like branes	space-like branes
reduction includes time	reduction does not include time
$\mathrm{d}s_D^2 = f^2(r)\mathrm{d}r^2 + g^2(r)g_{ab}\mathrm{d}x^a\mathrm{d}x^b$	$\mathrm{d}s_D^2 = -f^2(t)\mathrm{d}t^2 + g^2(t)g_{ab}\mathrm{d}x^a\mathrm{d}x^b$
pseudo-Riem. moduli space	Riem. moduli space
$\frac{G}{H^*}$ with H^* non-compact	$\frac{G}{H}$ with H compact
$\|\|\mathrm{v}\|\|^2 > 0, < 0, = 0$	$\|\|\mathrm{v}\|\|^2 > 0$
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The geodesic equations in Lax pair form

- Consider a symmetric space G/H (Riem. or pseudo-Riem.). The Cartan decomposition reads

$$\mathbb{G} = \mathbb{H} + \mathbb{K}\,,$$
$$[\mathbb{H},\mathbb{H}] \subset \mathbb{H}\,, \quad [\mathbb{H},\mathbb{K}] \subset \mathbb{K}\,, \quad [\mathbb{K},\mathbb{K}] \subset \mathbb{H}\,.$$

- Upon choosing a coset representative $\mathbb{L}(\phi^I(t))$, one can build the Maurer-Cartan form

$$\Omega = \mathbb{L}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L} = \dot{\phi}^I \mathbb{L}^{-1}\frac{\partial}{\partial \phi^I}\mathbb{L} = W + V\,,$$

with $W \in \mathbb{H}$, $V \in \mathbb{K}$.
- The scalar field action reads

$$S = \int \mathrm{d}t\, \mathrm{Tr}(VV) \propto \int \mathrm{d}t\, G_{IJ}(\phi)\dot{\phi}^I\dot{\phi}^J\,.$$

- Varying this action, one is led to the following equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}V = [V, W]\,.$$

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- The equation
$$\frac{\mathrm{d}}{\mathrm{d}t} V = [V, W] \, ,$$
constitutes the so-called Lax equation. It reproduces the geodesic equations as a matrix differential equation.
- Note : $V(t) = V(\phi(t), \dot{\phi}(t)), W(t) = W(\phi(t), \dot{\phi}(t))$.
- For symmetric spaces, one can work in *solvable gauge* :
$$\mathbb{L} = \exp b \, , \qquad b \in \text{Borel algebra} \, .$$
"L = exponential of upper triangular matrix".
- In solvable gauge
$$W = V_{>0} - V_{<0} \, .$$
- The Lax equation, with W obeying the latter equation, can be solved algorithmically, for generic initial condition $V(t = 0) = V_0$.

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Initial conditions

- In order to solve the Lax equation, one needs to specify an initial condition V_0. This is taken to be an arbitrary (constant) element of $\mathbb{K}$.
- In general :

$$\begin{aligned} \mathbb{H} &= \text{Span}\,\{E^\alpha + \theta(E^\alpha)\}\ , \\ \mathbb{K} &= \text{Span}\,\{H_i, (E^\alpha - \theta(E^\alpha)\}\ . \end{aligned}$$

with

$$\begin{aligned} \text{space}-\text{like branes}: \theta(E^\alpha) &= -E^{-\alpha} = -(E^\alpha)^T\ , \\ \text{time}-\text{like branes}: \theta(E^\alpha) &= -(-1)^{\beta_0(\alpha)}E^{-\alpha} = -(-1)^{\beta_0(\alpha)}(E^\alpha)^T\ . \end{aligned}$$

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- Strategy for parametrizing initial values : parametrize V_0 as

$$V_0 = h(\exp Q_N)h^{-1}\,, \qquad h \in H\,,$$

with Q_N the so-called *normal form*.

- Space-like branes : elements of $\mathbb{K}$ are either diagonal, or symmetric $\Rightarrow$ they can be diagonalized using H-transformations. The eigenvalues are moreover real.
- Time-like branes : elements of $\mathbb{K}$ are either diagonal, symmetric or anti-symmetric. The normal form is no longer diagonal. Generically (**Bergshoeff et al. : arXiv:0806.2310**)

$$Q_N \quad \in \quad \Big\{ \underbrace{\Big(\frac{\mathfrak{sl}(2,\mathbb{R})}{\mathfrak{so}(1,1)}\Big)^p}_{\text{complex eigval.}} \times \underbrace{\mathfrak{so}(1,1)^q}_{\text{real eigval.}} \Big\} \underbrace{\oplus \,\text{Nil}}_{\text{nilpotent el.}}$$

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The simplest example : $S\ell(2,\mathbb{R})$

- Generators, coset representative and Lax operator

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbb{L} = \mathrm{e}^{\chi(t)E}\mathrm{e}^{\frac{\phi(t)}{2}H}.$$

$$V = \begin{pmatrix} \frac{1}{2}\phi'[t] & \pm\frac{1}{2}e^{-\phi[t]}\chi'[t] \\ \frac{1}{2}e^{-\phi[t]}\chi'[t] & -\frac{1}{2}\phi'[t] \end{pmatrix}$$

- Space-like branes ($S\ell(2,\mathbb{R})/SO(2)$):

$$\mathbb{H} = \mathrm{Span}\left\{(E-E^T)\right\}, \qquad \mathbb{K} = \mathrm{Span}\left\{H, \frac{1}{\sqrt{2}}(E+E^T)\right\},$$

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We now have to distinguish three cases:

- $a^2 > b^2$: normal form is diagonal with 2 real eigenvalues : $\lambda_\pm = \pm\sqrt{a^2-b^2}$. Corresponds to geodesics with positive norm squared.
- $a^2 < b^2$: 2 complex eigenvalues $\lambda, \bar\lambda = \pm i\sqrt{a^2-b^2}$. Corresponds to geodesics with negative norm squared.
- $a^2 = b^2$:

$$V_0 \propto \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

This case is nilpotent of degree 2 : $V_0^2 = 0$. Corresponds to null geodesics.

The simplest example : $S\ell(2,\mathbb{R})$

- Time-like branes $(S\ell(2,\mathbb{R})/SO(1,1))$:

$$\mathbb{H} = \text{Span}\left\{(E+E^T)\right\}, \qquad \mathbb{K} = \text{Span}\left\{H, \frac{1}{\sqrt{2}}(E-E^T)\right\},$$
$$V_0 = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, \quad a, b \in \mathbb{R}$$

We now have to distinguish three cases:

- $a^2 > b^2$: normal form is diagonal with 2 real eigenvalues : $\lambda_\pm = \pm\sqrt{a^2-b^2}$. Corresponds to geodesics with positive norm squared.
- $a^2 < b^2$: 2 complex eigenvalues $\lambda, \bar{\lambda} = \pm i\sqrt{a^2-b^2}$. Corresponds to geodesics with negative norm squared.
- $a^2 = b^2$:

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- Mathematicians have developed an integration algorithm that solves the Lax equation (**Kodama et al. : solv-int/9505004, solv-int/9506005**).
- This algorithm is universal : works both for Riemannian and pseudo-Riemannian cosets. The result is a solution $V_{\text{sol}}(t)$ such that

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- The version of Kodama et al. only incorporates diagonalizable initial conditions (real/complex eigenvalues).
- We gave however an integration formula that works for generic initial conditions, so including the nilpotent cases (Chemissany et al. 2009). Result:

$$V_{pq} = V_{pq}(t, V_0) .$$

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- Proving an integration formula is one thing: establishing the complete inetgrability of the geodesic equations is a separate issue.
- Liouville Integrability: is the statement that there exist n functionally independent constant of motion $\mathcal{H}_i(Z)$ (hamiltonians):

$$\{\mathcal{H}_i, \mathcal{H}_j\} = 0$$

- The geodesic Lagrangian reads

$$\mathcal{L} = \frac{1}{2} g_{AB} Y^A Y^B = \frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j, \qquad V = Y^A K_A$$

- Phase space variable are denoted by $\{\phi^i, P_j\}$, thereby the geodesic Eqs take the form

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- Using the poisson bracket on the phase space and $Y^A = g^{AB} V_B{}^i P_i$ together with MC Eqs, we obtain

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- Noether Charge: Consider the following matrix

$$Q = \mathbb{L}(\tau)V(\tau)\mathbb{L}(\tau)^{-1}, \qquad \frac{dQ}{d\tau} = 0$$

The n components of Noether charge matrix defined by

$$Q_A \sim \mathrm{Tr}(QT_A)$$

- One can derive the following relations between Q_A and Y_A

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- Proof: Establishing Liouville integrability of the *first order problem*

$$\dot{Y}_A + \{\mathcal{H}, Y_A\} = 0$$

- Poisson bracket on the dual Lie algebra $Solv^*$ is degenerate, integrability implies the existence of symplectic foliation for which the hamiltonian flows are integrable on the symplectic leaves.
- Each leaf is nothing but the co-adjoint orbit of an element (Y^A) of $Solv^*$.
- Denote

$$\dim(coset) = n, \quad \dim(leaf) = 2h_O, \quad 2h_O = \mathrm{rank}(f_{AB}{}^C Y_C)$$

- We proved the existence of $(n - h_O)$ constants of motion in involution where:
  $h_O \to$ corresponds to Hamilt. in involution on symplectic leaf.
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Conclusion and Outlook

- We have established new insights into the solvability and integrability of the geodesic eqs following from reducing symmetric supergravities over the timelike direction.
- We have presented a recursive but closed formula for the coset representative describing a generic geodesic solution.
- Our results solve an open-standing question about the existence of a fake superpotential (Hamilton-Jacobi) for black hole solutions.
- Since Liouville Integrability implies HJ integrability we have proven the (local) existence of a fake superpotential for all stationary and spherically symmetric BH's.
- We have given the physical interpretation for most of the Hamiltonians, e.g. the polynomial constants keep track of the regularity and extremality of the solutions etc..
- We anticipate to investigate the Hamilt. in more involved models, such as the STU model.
- We believe that the n Hamilt. will provide a complete set of commuting observables for the quantum description of BH.

Conclusion and Outlook

- We have established new insights into the solvability and integrability of the geodesic eqs following from reducing symmetric supergravities over the timelike direction.
- We have presented a recursive but closed formula for the coset representative describing a generic geodesic solution.
- Our results solve an open-standing question about the existence of a fake superpotential (Hamilton-Jacobi) for black hole solutions.
- Since Liouville Integrability implies HJ integrability we have proven the (local) existence of a fake superpotential for all stationary and spherically symmetric BH's.
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