General polarization modes for the Rosen gravitational wave

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Brinkmann Form

Strong field gravitational waves are often represented in the Brinkmann† or Rosen†† forms.

The Brinkmann form for a general *pp* wave is given by

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + H(u,x,y)\,\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{1}$$

For which the only non-zero component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2}\left\{\partial_x^2 H(u,x,y) + \partial_y^2 H(u,x,y)\right\}. \tag{2}$$

Restricting ourselves to vacuum plane waves gives us the form:

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + \left\{[x^2 - y^2]\,H_+(u) + 2xy\,H_\times(u)\right\}\,\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{3}$$

So the $+$ and $\times$ polarizations have explicitly decoupled.

† H. W. Brinkmann, "Einstein spaces which are mapped conformally on each other", Mathematische Annalen **18** (1925)119. doi:10.1007/BF01208647.
†† A. Einstein and N. Rosen, "On gravitational waves", J. Franklin Inst. **223** (1937) 43.

Rosen form

The most general form of the Rosen metric is

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + g_{AB}(u)\,\mathrm{d}x^A\,\mathrm{d}x^B. \tag{4}$$

It is a standard result† that the only non-zero component of R_{ab} is:

$$R_{uu} = -\left\{ \frac{1}{2}\, g^{AB}\, g''_{AB} - \frac{1}{4}\, g^{AB}\, g'_{BC}\, g^{CD}\, g'_{DA} \right\}. \tag{5}$$

Though relatively compact, the matrix inversions implicit in raising the indices mean that this quantity is grossly nonlinear.

In particular, the $+$ and $\times$ linear polarizations do not decouple in any obvious way.

† Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard Herlt, *Exact Solutions of Einstein's Field Equations*. (Cambridge University Press, Cambridge, 2003), section 24.5.

Linear Polarization

Consider the strong-field gravity wave metric in the + linear polarization, so $g_{xy} = 0$.

The resulting metric can be written in the form

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + f^2(u)\,\mathrm{d}x^2 + g^2(u)\,\mathrm{d}y^2. \tag{6}$$

The only non-zero component of the Ricci tensor is:

$$R_{uu} = -\left\{\frac{f''}{f} + \frac{g''}{g}\right\}. \tag{7}$$

This can be put into a more tractable form if we write the metric as

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + S^2(u)\,\left\{e^{+X(u)}\,\mathrm{d}x^2 + e^{-X(u)}\mathrm{d}y^2\right\}, \tag{8}$$

then

$$R_{uu} = -\frac{1}{2}\left\{4\,\frac{S''}{S} + (X')^2\right\}. \tag{9}$$

Now in vacuum we have

$$X' = 2\sqrt{-S''/S}, \qquad \text{so that} \qquad X(u) = 2\int^u \sqrt{-S''/S}\,\mathrm{d}u. \tag{10}$$

Linear Polarization

So the Rosen form for the $+$ polarization is

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + S^2(u)\,\left\{\exp\left(2\int^u \sqrt{-S''/S}\,\mathrm{d}u\right)\,\mathrm{d}x^2 + \exp\left(-2\int^u \sqrt{-S''/S}\,\mathrm{d}u\right)\mathrm{d}y^2\right\}. \quad (11)$$

We can explicitly construct a $\times$ polarization by rotating the solution for the $+$ polarization by 45° which gives us

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + S^2(u)\,\left\{\cosh\left(2\int^u \sqrt{-S''/S}\,\mathrm{d}u\right)\,[\mathrm{d}x^2 + \mathrm{d}y^2] + 2\sinh\left(2\int^u \sqrt{-S''/S}\,\mathrm{d}u\right)\mathrm{d}x\,\mathrm{d}y\right\}. \quad (12)$$

Likewise we can form any linear polarization by a rotation in the xy plane.

Note here we have split g_{AB} into a unit determinant matrix of hyperbolic functions and an "envelope function" $S(u)$.

Arbitrary Polarization

Take an arbitrary, u dependent polarization, and consider the following metric ansatz:

$$\begin{aligned} \mathrm{d}s^2 \;&=\; -2\,\mathrm{d}u\,\mathrm{d}v + S^2(u)\;\Big\{[\cosh(X(u)) + \cos(\theta(u))\sinh(X(u))]\mathrm{d}x^2 \\ &\qquad + 2\sin(\theta(u))\sinh(X(u))\mathrm{d}x\,\mathrm{d}y \\ &\qquad + [\cosh(X(u)) - \cos(\theta(u))\sinh(X(u))]\mathrm{d}y^2\Big\}. \end{aligned} \tag{13}$$

This reduces to linear polarizations when $\theta(u)$ is a constant.
Now

$$R_{uu} = -\frac{1}{2}\left\{4\,\frac{S''}{S} + (X')^2 + \sinh^2(X(u))\;(\theta')^2\right\}. \tag{14}$$

The vacuum equations are simply

$$4\,\frac{S''}{S} + (X')^2 + \sinh^2(X(u))\;(\theta')^2 = 0. \tag{15}$$

Arbitrary Polarization

Introduce a dummy function $L(u)$ and split:

$$4\,\frac{S''}{S} + (L')^2 = 0, \tag{16}$$

$$(L')^2 = (X')^2 + \sinh^2(X(u))\,(\theta')^2. \tag{17}$$

Equation (16) is the equation that had to be solved for linear polarization. The second of these equations can be rewritten as

$$\mathrm{d}L^2 = \mathrm{d}X^2 + \sinh^2(X)\,\mathrm{d}\theta^2, \tag{18}$$

and is the statement that L can be interpreted as distance in the 2-dimensional hyperbolic plane H_2.

Thus if we pick an arbitrary curve in the (X, θ) plane and find its length, $L(u)$ then solve for $S(u)$ we have the solution for an arbitrary polarization.

Comparison to EM waves

Compare this situation with electromagnetic waves. A arbitrary polarization can be written as

$$\vec{E}(u) = E_x(u)\,\hat{x} + E_y(u)\,\hat{y}, \tag{19}$$

without further constraint, so could be viewed as a walk in the (E_x, E_y) plane. Or alternatively, with a coordinate transform,

$$\vec{E}(u) = E(u)\cos\theta(u)\,\hat{x} + E(u)\sin\theta(u)\,\hat{y}. \tag{20}$$

So an arbitrary polarization can be viewed as a random walk in the (E, θ) plane, where the (E, θ) plane has the natural Euclidean metric

$$\mathrm{d}L^2 = \mathrm{d}E^2 + E^2\,\mathrm{d}\theta^2 \tag{21}$$

In contrast we are now dealing with a walk in the hyperbolic plane H_2.

We also have a remaining equation to solve for $S(u)$ due to the inherent nonlinearities in strong field gravity

Circular Polarization

As an example, consider circular polarization in the Rosen form.

For this we want $\theta(u)$ to progress linearly with a constant distortion, $X(u)$

$$\theta(u) = \Omega_0\ u; \qquad X(u) = X_0. \tag{22}$$

So

$$\begin{aligned} \mathrm{d}s^2 \;\; = \;\; & -2\,\mathrm{d}u\,\mathrm{d}v + S^2(u)\ \Big\{[\cosh(X_0) + \cos(\Omega_0\ u)\sinh(X_0)]\mathrm{d}x^2 \\ & + 2\sin(\Omega_0 u)\sinh(X_0)\mathrm{d}x\,\mathrm{d}y \\ & + [\cosh(X_0) - \cos(\Omega_0 u)\sinh(X_0)]\mathrm{d}y^2\Big\}\,. \end{aligned} \tag{23}$$

Circular Polarization

The only nontrivial component of the Ricci tensor is then

$$R_{uu} = -\frac{1}{2}\left\{4\,\frac{S''}{S} + \sinh^2(X_0)\,\Omega_0^2\right\}. \tag{24}$$

The vacuum equations are can be solved for

$$S(u) = S_0\;\cos\left\{\frac{\sinh(X_0)\,\Omega_0\,(u-u_0)}{2}\right\}. \tag{25}$$

So we have fully solved circular polarization in the Rosen form

This agrees with the limit of weak field gravity, which corresponds to $X_0 \ll 1$ so $S \approx S_0$ and

$$\mathrm{d}s^2 \approx -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 + X_0\;\left\{\cos(\Omega_0\;u)[\mathrm{d}x^2 - \mathrm{d}y^2] + 2\sin(\Omega_0\;u)\,\mathrm{d}x\;\mathrm{d}y\right\}. \tag{26}$$

Rosen form in arbitrary dimensions

Consider a *pp* wave in Rosen form with $d_\perp$ dimensions perpendicular to the direction of travel. Again

$$R_{uu} = -\left\{ \frac{1}{2}\, g^{AB}\, g''_{AB} - \frac{1}{4}\, g^{AB}\, g'_{BC}\, g^{CD}\, g'_{DA} \right\}. \tag{27}$$

Split $g_{AB}(u)$ into a "envelope" $S(u)$ and a unit determinant $\hat{g}_{AB}(u)$

$$g_{AB}(u) = S^2(u)\, \hat{g}_{AB}(u), \tag{28}$$

Calculating the various components in the Ricci tensor, using the relation

$$[\hat{g}^{AB}\, \hat{g}'_{AB}] = 0, \tag{29}$$

And differentiating,

$$[\hat{g}^{AB}\, \hat{g}''_{AB}] - [\hat{g}^{AB}\, \hat{g}'_{BC}\, \hat{g}^{CD}\, \hat{g}'_{DA}] = 0. \tag{30}$$

It is found that

$$R_{uu} = -d_\perp\, \frac{S''}{S} - \frac{1}{2}\, [\hat{g}^{AB}\, \hat{g}'_{BC}\, \hat{g}^{CD}\, \hat{g}'_{DA}]. \tag{31}$$

By splitting the metric we have simplified the vacuum equations, decoupling the parts depending on the "envelope" and the "direction of oscillation".

Rosen form in arbitrary dimensions

Consider the set $SS(\mathbb{R}, d_\perp)$ of unit determinant real symmetric matrices and the Riemannian metric

$$\mathrm{d}L^2 = Tr\left\{[\hat{g}]^{-1}\,\mathrm{d}[\hat{g}]\;[\hat{g}]^{-1}\,\mathrm{d}[\hat{g}]\right\}. \tag{32}$$

Then

$$R_{uu} = -\frac{1}{2}\left\{2d_\perp\,\frac{S''(u)}{S(u)} + \left(\frac{\mathrm{d}L}{\mathrm{d}u}\right)^2\right\}. \tag{33}$$

The vacuum Einstein equations reduce to

$$\frac{\mathrm{d}L}{du} = \sqrt{-2d_\perp\,\frac{S''}{S}}; \qquad L(u) = \int^u \sqrt{-2d_\perp\,\frac{S''}{S}}\,du. \tag{34}$$

An arbitrary polarization of a vacuum Rosen wave is a random walk in $SS(\mathbb{R}, d_\perp)$, with distance along the walk $L(u)$ being related to the envelope function $S(u)$ as above.

Summary

- Arbitrary polarizations, while trivial in the Brinkmann form, pose a difficulty in the Rosen form.
- We have made progress on this by splitting the relevant equations into an "envelope" function and a unit determinant polarization matrix.
- The vacuum equations reduce to a differential equation regarding the envelope and a random walk in polarization space.
- This has been generalized to arbitrary dimensions.
- Based on this we can construct arbitrary polarisation states, and in particular have constructed a circularly polarized strong field wave in Rosen form.