Double shell stars as source of the Kerr metric in the CMMR approximation

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Outline

1. The CMMR approximation method and some previous applications
2. Double shell source: Stating the problem
3. Building the global solution
4. Its multipole moments and Kerr

CMMR method and applications

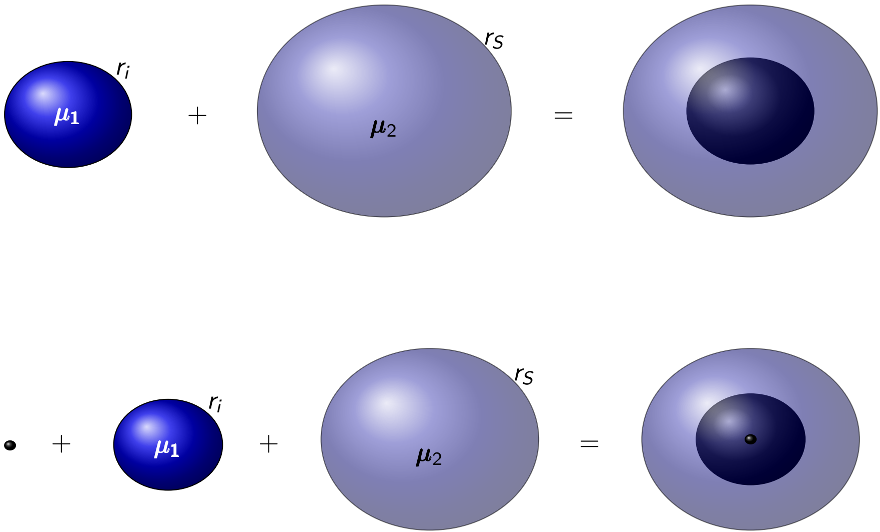
The method

- It is a double perturbative method
  - Post-Minkowskian: $\mathbf{g}(\lambda, \Omega) = \boldsymbol{\eta} + \mathbf{h}(\lambda, \Omega)$ with λ a parameter related with the strengh of the field.
  - Slow rotation: A parameter related with the rotation speed, Ω.
- We can use to build global axisymmetric stationary solutions
- It is analytical and then can be useful for theorists.

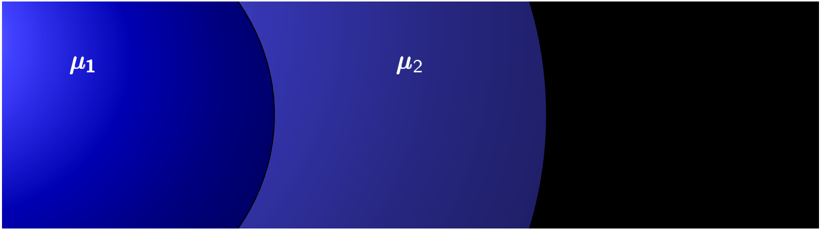
Some previous uses

- Studied the viability as Kerr sources of
  - Compact ball of constant density perfect fluid
  - Polytropic perfect fluid ball
  - Compact ball of perfect fluid with equation of state $\mu + (1 - n)p = \mu_0$
- Confirmed that there is no assymptotically flat exterior for Wahlquist metric.

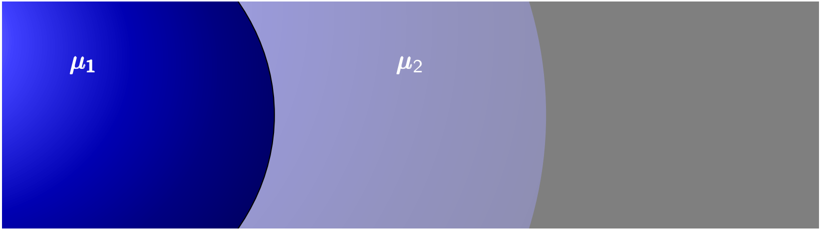
Forming a picture



Characteristics of the source and vacuum

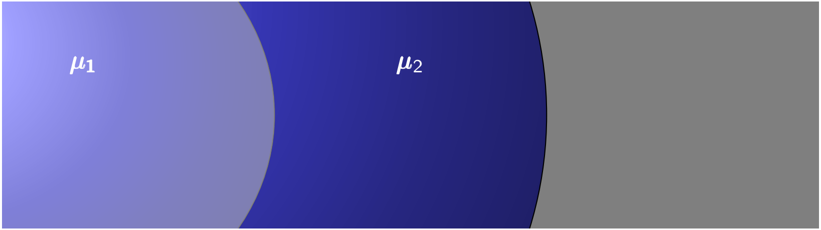


Characteristics of the source and vacuum



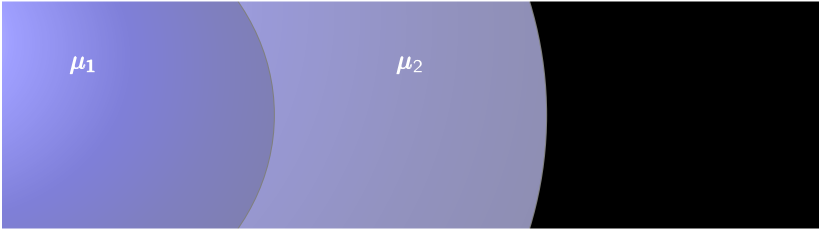
- Stationary: ξ a timelike Killing field
- Axisymmetry: ζ a closed-orbits spacelike Killing field
- Equatorial symmetry
- Perfect fluid. No convective motion: $u^\alpha = \psi\,(\xi^\alpha + \omega\,\zeta^\alpha)$
- Equation of state: $\mu_1 + (1-n)p = \epsilon_1$
- Rigid motion ($\omega = \text{const}$)
- Surface: $r_I = r_i\left[1 + S_i\,\Omega^2 P_2(\cos\theta)\right]$

Characteristics of the source and vacuum



- Stationary and axisymmetric + equatorial symmetry
- Perfect fluid. No convective motion
- Equation of state: $\mu_2 + (1-n)p = \epsilon_2$
- Rigid motion ($\omega =$ const)
- Surface: $r_\Sigma = r_s \left[1 + S_s\, \Omega^2 P_2(\cos\theta)\right]$

Characteristics of the source and vacuum



- Stationary and axisymmetric + equatorial symmetry
- Vacuum
- Assymptotically flat

Double shell: General solutions

Vacuum solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + 2\sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} \left(\mathbf{T}_n + \mathbf{D}_n\right) + 2\sum_{n=1}^{\infty} \frac{J_n}{r^{n+1}} \mathbf{Z}_n + \sum_{n=0,2} \frac{A_n}{r^{n+3}} \mathbf{E}_{n+2} + \frac{B_2}{r^3} \mathbf{F}_2 \ .$$

Core solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + \sum_{n=0}^{\infty} m_n \, r^n \left(\mathbf{T}_n + \mathbf{D}_n\right) + \sum_{n=1}^{\infty} j_n \, r^n \, \mathbf{Z}_n + \sum_{n=0}^{\infty} a_n r^n \mathbf{E^*}_n + \sum_{n=0}^{\infty} b_n \, r^n \, \mathbf{F^*}_n \, .$$

Shell solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + \sum_{n=0}^{\infty} \tilde{m}_n \, r^n \left(\mathbf{T}_n + \mathbf{D}_n\right) + \sum_{n=1}^{\infty} \tilde{j}_n \, r^n \, \mathbf{Z}_n + \sum_{n=0}^{\infty} \tilde{a}_n r^n \mathbf{E^*}_n + \sum_{n=0}^{\infty} \tilde{b}_n \, r^n \, \mathbf{F^*}_n$$
$$+ 2\sum_{n=0}^{\infty} \frac{\tilde{M}_n}{r^{n+1}} \left(\mathbf{T}_n + \mathbf{D}_n\right) + 2\sum_{n=1}^{\infty} \frac{\tilde{J}_n}{r^{n+1}} \mathbf{Z}_n + \sum_{n=0,2} \frac{\tilde{A}_n}{r^{n+3}} \mathbf{E}_{n+2} + \frac{\tilde{B}_2}{r^3} \mathbf{F}_2 \ .$$

Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{array}{ll} M_n \to \lambda \Omega^n \dfrac{\bar{M}_n}{r_s^n}\,, & J_n \to \lambda^{\frac{3}{2}} \Omega^n \dfrac{\bar{J}_n}{r_s^n}\,, \\ A_n \to \lambda \Omega^n \dfrac{\bar{A}_n}{r_s^{n+3}}\,, & B_2 \to \lambda \Omega^2 \dfrac{\bar{B}_n}{r_s^3}\,. \end{array}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda \bar{X}^{(2)} + \ldots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{m}_0^{(1)} = \frac{3\left({r_i}^2(\epsilon_1 - \epsilon_2) + {r_s}^2 \epsilon_2\right)}{{r_0}^2},$$

$$\bar{m}_2^{(1)} = \frac{2\left(5{r_i}^3{r_s}^2(\epsilon_1-\epsilon_2)^2 + {r_s}^5(7\epsilon_1 - 2\epsilon_2)\epsilon_2 + 3{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2\right)}{9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2 + \epsilon_1\epsilon_2 - 3{\epsilon_2}^2\right)}$$

$$\bar{j}_1^{(1)} = \frac{2\left({r_i}^2(\epsilon_1 - \epsilon_2) + {r_s}^2 \epsilon_2\right)}{{r_0}^2} + \Omega^2 \times$$

$$\times\ \frac{-4{r_s}^2\left(5{r_i}^2{r_s}^3(\epsilon_1-\epsilon_2)\epsilon_2 + {r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - {r_i}^5\left(5{\epsilon_1}^2 - 7\epsilon_1\epsilon_2 + 2{\epsilon_2}^2\right)\right)}{3{r_0}^2\left(9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2+\epsilon_1\epsilon_2 - 3{\epsilon_2}^2\right)\right)},$$

Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{array}{ll} M_n \to \lambda\Omega^n \dfrac{\bar{M}_n}{r_s^n}\,, & J_n \to \lambda^{\frac{3}{2}}\Omega^n \dfrac{\bar{J}_n}{r_s^n}\,, \\ A_n \to \lambda\Omega^n \dfrac{\bar{A}_n}{r_s^{n+3}}\,, & B_2 \to \lambda\Omega^2 \dfrac{\bar{B}_n}{r_s^3}\,. \end{array}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda\bar{X}^{(2)} + \dots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{J}_3^{(1)} = \frac{4\left(5{r_i}^3{r_s}^2(\epsilon_1-\epsilon_2)^2 + {r_s}^5(7\epsilon_1-2\epsilon_2)\epsilon_2 + 3{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2\right)}{7\left(9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2+\epsilon_1\epsilon_2-3{\epsilon_2}^2\right)\right)},$$

$$\bar{M}_0^{(1)} = \frac{{r_i}^3(\epsilon_1-\epsilon_2) + {r_s}^3\epsilon_2}{{r_0}^3},$$

$$\bar{J}_1^{(1)} = \frac{2\left({r_i}^5(\epsilon_1-\epsilon_2) + {r_s}^5\epsilon_2\right)}{5{r_0}^5} + \Omega^2 \times$$

$$\times\ \frac{-2{r_s}^2\left(5{r_i}^8(\epsilon_1-\epsilon_2)^2 - 8{r_i}^5{r_s}^3(\epsilon_1-\epsilon_2)\epsilon_2 - {r_s}^8\epsilon_2(2\epsilon_1+3\epsilon_2)\right)}{3{r_0}^5\left(9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2+\epsilon_1\epsilon_2-3{\epsilon_2}^2\right)\right)},$$

Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{array}{ll} M_n \to \lambda\Omega^n \dfrac{\bar{M}_n}{r_s^n}\,, & J_n \to \lambda^{\frac{3}{2}}\Omega^n \dfrac{\bar{J}_n}{r_s^n}\,, \\ A_n \to \lambda\Omega^n \dfrac{\bar{A}_n}{r_s^{n+3}}\,, & B_2 \to \lambda\Omega^2 \dfrac{\bar{B}_n}{r_s^3}\,. \end{array}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda\bar{X}^{(2)} + \dots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{M}_2^{(1)} = \frac{{r_s}^2\left(5{r_i}^8(\epsilon_1-\epsilon_2)^2 + 8{r_i}^5{r_s}^3(\epsilon_1-\epsilon_2)\epsilon_2 + {r_s}^8\epsilon_2(2\epsilon_1+3\epsilon_2)\right)}{{r_0}^5\left(9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2+\epsilon_1\epsilon_2-3{\epsilon_2}^2\right)\right)},$$

$$\begin{aligned} \bar{J}_3^{(1)} = {} & \frac{1}{7{r_0}^7\left(9{r_i}^5(\epsilon_1-\epsilon_2)\epsilon_2 - 2{r_s}^5\epsilon_2(2\epsilon_1+3\epsilon_2) - 5{r_i}^3{r_s}^2\left(2{\epsilon_1}^2+\epsilon_1\epsilon_2-3{\epsilon_2}^2\right)\right)}\times \\ & \times\Big[2{r_s}^2\left(5{r_i}^{10}(\epsilon_1-\epsilon_2)^2 + 5{r_i}^7{r_s}^3(\epsilon_1-\epsilon_2)\epsilon_2\right. \\ & \qquad \left. + 3{r_i}^5{r_s}^5(\epsilon_1-\epsilon_2)\epsilon_2 + {r_s}^{10}\epsilon_2(2\epsilon_1+3\epsilon_2)\right)\Big], \end{aligned}$$

Comparing with Kerr

Kerr multipole moments are given by: $M_n^{Kerr} = m(ia)^n$
If we relate them with ours we have

$$M_0^{Kerr} = m \xrightarrow{ours} \lambda r_s \bar{M}_0$$

$$M_1^{Kerr} \equiv J_1^{Kerr} = ma \xrightarrow{ours} \lambda^{3/2}\Omega r_s^2 \bar{J}_1$$

$$M_2^{Kerr} = -ma^2 \xrightarrow{ours} \lambda\Omega^2 r_s^3 \bar{M}_2$$

$$M_3^{Kerr} \equiv J_3^{Kerr} = ma^3 \xrightarrow{ours} \lambda^{3/2}\Omega^3 r_s^4 \bar{J}_3$$

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$$M_3^{Kerr} \equiv J_3^{Kerr} = ma^3 \xrightarrow{ours} \lambda^{3/2}\Omega^3 r_s^4 \bar{J}_3$$

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^2 = -\frac{(\lambda^{3/2}\Omega r_s^2 \bar{J}_1)^2}{\lambda r_s \bar{M}_0} = -\lambda^2\Omega^2 r_s^3 \frac{\bar{J}_1^2}{\bar{M}_0}$$

$$M_2^{\text{Kerr}} = -ma^2 = \lambda\Omega^2 r_s^3(\bar{M}_2^{(1)} + \lambda \bar{M}_2^{(2)} + \dots) \quad \longrightarrow \bar{M}_2^{(1)} \text{ must be zero}$$

Comparing with Kerr

Kerr multipole moments are given by: $M_n^{Kerr} = m(ia)^n$
If we relate them with ours we have

$$M_0^{Kerr} = m \xrightarrow{ours} \lambda r_s \bar{M}_0$$
$$M_1^{Kerr} \equiv J_1^{Kerr} = ma \xrightarrow{ours} \lambda^{3/2}\Omega r_s^2 \bar{J}_1$$
$$M_2^{Kerr} = -ma^2 \xrightarrow{ours} \lambda\Omega^2 r_s^3 \bar{M}_2$$
$$M_3^{Kerr} \equiv J_3^{Kerr} = ma^3 \xrightarrow{ours} \lambda^{3/2}\Omega^3 r_s^4 \bar{J}_3$$

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^3 = -\lambda^{5/2}\Omega^3 r_s^4 \frac{\bar{J}_1^3}{\bar{M}_0^2}$$

$$J_3 = \lambda^{3/2}\Omega^3 r_s^4 (\bar{J}_3^{(1)} + \lambda \bar{J}_3^{(2)} + \dots) \quad \longrightarrow \bar{J}_3^{(1)} \text{ must be zero}$$

$$\bar{J}_3^{(1)} = \bar{M}_2^{(1)} = 0 \iff \epsilon_1 = \epsilon_2 = 0$$

Double Shell + singularity: General solutions

Vacuum solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + 2\sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} \left(\mathbf{T}_n + \mathbf{D}_n\right) + 2\sum_{n=1}^{\infty} \frac{J_n}{r^{n+1}} \mathbf{Z}_n + \sum_{n=0,2} \frac{A_n}{r^{n+3}} \mathbf{E}_{n+2} + \frac{B_2}{r^3} \mathbf{F}_2 \ .$$

Core solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + \sum_{n=0}^{\infty} \left(m_n\, r^n + \frac{A}{r} \right) \left(\mathbf{T}_n + \mathbf{D}_n\right) + \sum_{n=1}^{\infty} j_n\, r^n\, \mathbf{Z}_n + \sum_{n=0}^{\infty} a_n r^n \mathbf{E^*}_n + \sum_{n=0}^{\infty} b_n\, r^n\, \mathbf{F^*}_n \, .$$

Shell solution

$$\mathbf{h} = \mathbf{h}_{\rm inh} + \sum_{n=0}^{\infty} m_n\, r^n \left(\mathbf{T}_n + \mathbf{D}_n\right) + \sum_{n=1}^{\infty} j_n\, r^n\, \mathbf{Z}_n + \sum_{n=0}^{\infty} a_n r^n \mathbf{E^*}_n + \sum_{n=0}^{\infty} b_n\, r^n\, \mathbf{F^*}_n$$
$$+ 2\sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} \left(\mathbf{T}_n + \mathbf{D}_n\right) + 2\sum_{n=1}^{\infty} \frac{J_n}{r^{n+1}} \mathbf{Z}_n + \sum_{n=0,2} \frac{A_n}{r^{n+3}} \mathbf{E}_{n+2} + \frac{B_2}{r^3} \mathbf{F}_2 \ .$$

Double shell + singularity: Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{array}{ll} M_n \to \lambda \Omega^n \dfrac{\bar{M}_n}{r_s^n}\,, & J_n \to \lambda^{\frac{3}{2}} \Omega^n \dfrac{\bar{J}_n}{r_s^n}\,, \\ A_n \to \lambda \Omega^n \dfrac{\bar{A}_n}{r_s^{n+3}}\,, & B_2 \to \lambda \Omega^2 \dfrac{\bar{B}_n}{r_s^3}\,. \end{array}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda \bar{X}^{(2)} + \dots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{m}_0^{(1)} = \frac{3\left(r_i{}^2(\epsilon_1 - \epsilon_2) + r_s{}^2\epsilon_2\right)}{r_0{}^2},$$

$$\bar{m}_2^{(1)} = \frac{2r_i{}^8(\epsilon_1 - \epsilon_2) + 20{\color{red}A}r_0{}^6 r_s{}^2\epsilon_2 + 20 r_0{}^3 r_i{}^3 r_s{}^2 \epsilon_1\epsilon_2}{3r_i{}^8(\epsilon_1 - \epsilon_2)},$$

$$\bar{j}_1^{(1)} = \frac{5r_0{}^5 r_s{}^5 + 30 r_0{}^8 r_i{}^2(\epsilon_1 - \epsilon_2) + 2r_i{}^5 r_s{}^5(\epsilon_1 - \epsilon_2)}{15 r_0{}^{10}} + \Omega^2 \frac{-4r_s{}^2\left(r_i{}^8(\epsilon_1 - \epsilon_2) + 10 r_0{}^8\epsilon_2\right)}{15 r_0{}^7 r_i{}^3},$$

Double shell + singularity: Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{aligned} M_n &\to \lambda\Omega^n \frac{\bar{M}_n}{r_s^n}\,, \quad & J_n &\to \lambda^{\frac{3}{2}}\Omega^n \frac{\bar{J}_n}{r_s^n}\,, \\ A_n &\to \lambda\Omega^n \frac{\bar{A}_n}{r_s^{n+3}}\,, \quad & B_2 &\to \lambda\Omega^2 \frac{\bar{B}_n}{r_s^3}\,. \end{aligned}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda\bar{X}^{(2)} + \ldots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{J}_3^{(1)} = \frac{4{r_i}^{10}(\epsilon_1 - \epsilon_2) + 40{r_0}^8{r_s}^2\epsilon_2}{35{r_0}^5{r_i}^5},$$

$$\bar{M}_0^{(1)} = \frac{{r_s}^3\epsilon_2}{{r_0}^3}\,,$$

$$\bar{J}_1^{(1)} = \frac{2{r_s}^5\epsilon_2}{5{r_0}^5} + \Omega^2\frac{-20{r_0}^8{r_s}^2\epsilon_2 + 2{r_i}^5{r_s}^5(-\epsilon_1 + \epsilon_2)}{15{r_0}^{10}},$$

Double shell + singularity: Matching and results

After parametrizing the multipole moments such that

$$\left.\begin{aligned} M_n \to \lambda\Omega^n \frac{\bar{M}_n}{r_s^n}, \quad & J_n \to \lambda^{\frac{3}{2}}\Omega^n \frac{\bar{J}_n}{r_s^n}, \\ A_n \to \lambda\Omega^n \frac{\bar{A}_n}{r_s^{n+3}}, \quad & B_2 \to \lambda\Omega^2 \frac{\bar{B}_n}{r_s^3}. \end{aligned}\right\} \quad \text{where } \bar{X} = \bar{X}^{(1)} + \lambda\bar{X}^{(2)} + \dots$$

and imposing the Lichnerowicz conditions $[g_{\alpha\beta}]_\Sigma = [\partial_\alpha g_{\beta\gamma}]_\Sigma = 0$, we get

$$\bar{M}_2^{(1)} = \frac{{r_s}^5\left({r_0}^5 + {r_i}^5(\epsilon_1 - \epsilon_2)\right)}{3{r_0}^{10}},$$

$$\bar{J}_3^{(1)} = \frac{2{r_i}^5{r_s}^7(\epsilon_1 - \epsilon_2) + 20{r_0}^8{r_i}^2{r_s}^2\epsilon_2}{35{r_0}^{12}},$$

Double shell + singularity: Comparing with Kerr

Adjusting M_2

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^2 = -\frac{(\lambda^{3/2}\Omega r_s^2 \bar{J}_1)^2}{\lambda r_s \bar{M}_0} = -\lambda^2\Omega^2 r_s^3 \frac{\bar{J}_1^2}{\bar{M}_0}$$

$$M_2 = \lambda\Omega^2 r_s^3(\bar{M}_2^{(1)} + \lambda\bar{M}_2^{(2)} + \dots) \quad \longrightarrow \bar{M}_2^{(1)} \text{ must be zero}$$

In this case, $\bar{M}_2^{(1)} = \dfrac{{r_s}^5\left({r_0}^5 + {r_i}^5(\epsilon_1 - \epsilon_2)\right)}{3{r_0}^{10}}$.

Double shell + singularity: Comparing with Kerr

Adjusting M_2

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^2 = -\frac{(\lambda^{3/2}\Omega r_s^2 \bar{J}_1)^2}{\lambda r_s \bar{M}_0} = -\lambda^2\Omega^2 r_s^3 \frac{\bar{J}_1^2}{\bar{M}_0}$$

$$M_2 = \lambda\Omega^2 r_s^3(\bar{M}_2^{(1)} + \lambda\bar{M}_2^{(2)} + \dots) \quad \longrightarrow \bar{M}_2^{(1)} \text{ must be zero}$$

In this case, $\bar{M}_2^{(1)} = \dfrac{{r_s}^5\left({r_0}^5 + {r_i}^5(\epsilon_1 - \epsilon_2)\right)}{3{r_0}^{10}}$.

Adjusting J_3

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^3 = -\lambda^{5/2}\Omega^3 r_s^4 \frac{\bar{J}_1^3}{\bar{M}_0^2}$$

$$J_3 = \lambda^{3/2}\Omega^3 r_s^4(\bar{J}_3^{(1)} + \lambda\bar{J}_3^{(2)} + \dots) \quad \longrightarrow \bar{J}_3^{(1)} \text{ must be zero}$$

Here, $\bar{J}_3^{(1)} = \dfrac{2{r_i}^5{r_s}^7(\epsilon_1 - \epsilon_2) + 20{r_0}^8{r_i}^2{r_s}^2\epsilon_2}{35{r_0}^{12}}$.

Double shell + singularity: Comparing with Kerr

Solving

$$\left.\begin{aligned} {r_s}^5\left({r_0}^5+{r_i}^5(\epsilon_1-\epsilon_2)\right)=0 \\ 2{r_i}^5{r_s}^7(\epsilon_1-\epsilon_2)+20{r_0}^8{r_i}^2{r_s}^2\epsilon_2=0 \end{aligned}\right\} \longrightarrow \begin{aligned} \epsilon_1 &= -\frac{10{r_0}^8-{r_i}^3{r_s}^5}{10{r_0}^3{r_i}^5}, \\ \epsilon_2 &= \frac{{r_s}^5}{10{r_0}^3{r_i}^2} \end{aligned}$$

so that the double shell + singularity configuration is compatible with Kerr to this order

Adjusting J_3

$$\left.\begin{aligned} M_0^{\text{Kerr}} &= m = \lambda r_s \bar{M}_0 \\ J_1^{\text{Kerr}} &= ma = \lambda^{3/2}\Omega r_s^2 \bar{J}_1 \end{aligned}\right\} \quad -ma^3 = -\lambda^{5/2}\Omega^3 r_s^4 \frac{\bar{J}_1^3}{\bar{M}_0^2}$$

$$J_3 = \lambda^{3/2}\Omega^3 r_s^4(\bar{J}_3^{(1)} + \lambda \bar{J}_3^{(2)} + \dots) \quad \longrightarrow \bar{J}_3^{(1)} \text{ must be zero}$$

Here, $\bar{J}_3^{(1)} = \dfrac{2{r_i}^5{r_s}^7(\epsilon_1-\epsilon_2)+20{r_0}^8{r_i}^2{r_s}^2\epsilon_2}{35{r_0}^{12}}$.

Remarks

1. We have used the CMMR approximation method to build two double shell metrics
2. The first one is completely regular and cannot be a source of the Kerr spacetime
3. The second one has a singularity in the origin, and to the order computed, is compatible with a Kerr exterior
4. We need and plan to compute extra orders to check if this behaviour goes on and if it would be necessary to add extra singularities or additional density discontinuities.