Axially symmetric spacetimes: numerical and analytical perspectives

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Symmetries and Einstein equations

- The presence of a symmetry reduces the degrees of freedom of Einstein equations and hence it simplifies considerably its analysis.
- In vacuum, due to Birkoff's theorem, spherical symmetry has no dynamic.
- For isolated systems, the next possible model with symmetries are axially symmetric spacetimes. No additional symmetry can be imposed to the spacetime if we want to keep the gravitational radiation and a complete null infinity.
- Axial symmetry is the only symmetry we can impose to gravitational waves emitted by isolated systems.

Physical models in axial symmetry (strong field regime)

- Head-on collisions of two black holes.
- Rotating, non-stationary, black holes.
- Formation of black holes: weak cosmic censorship.
- Critical collapse of gravitational waves.

These models can be studied for **pure vacuum axially symmetric waves**.

Two relevant open problems (long term)

- The stability of the Kerr black hole in axial symmetry (Mathematical).
- Critical collapse of axially symmetric gravitational waves (Numerical).

Axial symmetry

$(\mathcal{V}, g_{\mu\nu})$: spacetime.
${}^{(4)}\mathcal{R}_{\mu\nu} = 0$ Einstein vacuum equations.
η^μ Killing vector (symmetry), $\bar{\nabla}_{(\nu}\eta_{\mu)} = 0$.
We define the square of the norm and the twist of η^μ, respectively, by

$$\eta = \eta^\mu \eta^\nu g_{\mu\nu}, \quad \omega_\mu = \epsilon_{\mu\nu\gamma\alpha}\eta^\nu \bar{\nabla}^\gamma \eta^\alpha.$$

Using the vacuum equations we obtain

$$\omega_\mu = \bar{\nabla}_\mu \omega.$$

$(\eta,\, \omega)$: dynamical degree of freedom of the gravitational waves.

Symmetry reduction

$\mathcal{N}$: 3-manifold defined as the collection of all trajectories of η^μ.
We define the Lorentzian metric $h_{\mu\nu}$ on $\mathcal{N}$ by

$$\eta g_{\mu\nu} = h_{\mu\nu} + \eta_\mu \eta_\nu .$$

We have rescaled the intrinsic metric $h_{\mu\nu}$ by a conformal factor η.
Einstein equations can be written as follows:

$$\nabla_a \nabla^a \eta = \frac{1}{\eta} \left(\nabla^a \eta \nabla_a \eta - \nabla^a \omega \nabla_a \omega \right),$$

$$\nabla_a \nabla^a \omega = \frac{2}{\eta} \nabla^a \omega \nabla_a \eta,$$

$$^{(3)}\mathcal{R}_{ab} = \frac{1}{2\eta} \left(\nabla_a \eta \nabla_b \eta + \nabla_a \omega \nabla_b \omega \right).$$

where ∇_a and $^{(3)}\mathcal{R}_{ab}$ are the connexion and the Ricci tensor of h_{ab}.

Gravitational Radiation=Effective Matter Sources

- We have reduced (a la Kaluza-Klein) Einstein vacuum equations in 4-dimensions to a system of equations intrinsic to $(\mathcal{N}, h_{ab})$.
- These equations take the form of 3-dimensional Einstein equations coupled to "effective" matter sources determined by (η, ω).
- Since in 3-dimensions there is no radiation (the Weyl tensor is zero), these sources represent the true gravitational degree of freedom that have descended from 4-dimensions to appear as "matter" in 3-dimensions.

Difficulties in axial symmetry

- To take advantage of the symmetry an adapted coordinate system (reduction) should be used: the norm η of the axial Killing vector vanished at the axis, and hence the reduced equations are formally singular there.
- **In fact, it can be argued that this singular behavior near the axis is so complicated that the axially symmetric case is as hard as the full general case.**

Advantages of axial symmetry

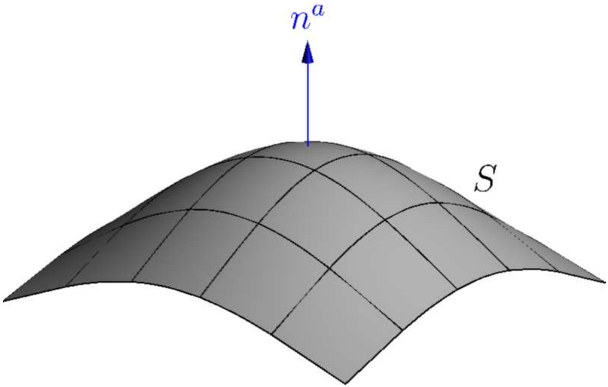
1. Less computationally expensive: only two effective spatial dimensions. The number of equations and variables are reduced.
2. Conservation of angular momentum: the angular momentum is a quasilocal conserved quantity in axial symmetry (Komar integral of the Killing vector).
   Axially symmetric gravitational waves do not carry angular momentum. **No Penrose process, no superradiant scattering.**
3. The mass integral formula: the total mass can be written as a positive definite volume integral, as we will see in the following.

Numerical Relativity → Mathematical Relativity

- The difficulty introduced by the singular behavior at the axis is so severe that until recently axially symmetric spacetimes have not been studied in detail even using numerical techniques.
- At the axis we have singular boundary conditions. In **Garfinkle, Duncan** 01; **Choptuik, Hirschmann, Liebling, Pretorius** 03; **Rinne** 08; this kind of singular boundary conditions have been successfully implemented numerically.

2+1 decomposition

We make a $2+1$ decomposition of the 3-dimensional, Lorenzian, metric h_{ab}.



q_{AB}: intrinsic metric on S.
χ_{AB}: second fundamental form of S.

Maximal isothermal gauge

Lapse: we impose the maximal condition on the surfaces

$$\chi = 0.$$

Shift: we require that the intrinsic metric q_{AB} has the following form

$$q_{AB} = e^{2u}\delta_{AB},$$

where δ_{AB} is the flat metric in two dimensions.
These conditions imply elliptic equations for lapse and shift.

The mass integral formula

The gauge fixed a coordinate system (t, ρ, z). We define σ by

$$\eta = \rho^2 e^{\sigma}.$$

The mass is given by

$$m = \frac{1}{16} \int_S \epsilon \, \rho \, d\rho dz,$$

where

$$\epsilon = \frac{e^{2u}}{\eta^2} \left(\eta'^2 + \omega'^2 \right) + 2e^{2u} \chi^{AB} \chi_{AB} + |\partial \sigma|^2 + \frac{|\partial \omega|^2}{\eta^2},$$

and

$$\eta' = n^a \nabla_a \eta.$$

The density ϵ is positive and regular at the axis.

Mass conservation

There exist a unique solution of the maximal-isothermal gauge equations and the mass integral given above is conserved along the evolution, that is

$$\frac{dm}{dt} = 0.$$

The role of the mass in the evolution

- In any physical theory conserved quantities (in particular, conserved energies) are very important to control the evolution of the system.
- However, in General Relativity, the conserved mass appears as a boundary integral and not as a volume integral (as, for example, in the wave equation).
- Hence it is not possible to relate the mass with any norm of the fields to control the evolution of them (for the wave equation the energy is precisely the appropriate norm of the solutions).
- Axially symmetric systems represent a remarkable exception.

Extreme Kerr as a minimum of the mass

Extreme Kerr is an absolute minimum of the mass for all regular axially symmetric data with fixed angular momentum. It follows that the inequality (S. D. 06)

$$m \geq \sqrt{|J|},$$

holds for axially symmetric (non-stationary) black holes. With equality if and only if the data are extreme Kerr. (This inequality has been generalized to **include charge** by Chruściel–Lopes Costa 10)

This suggests an stability property of extreme Kerr in axial symmetry.

Mathematical Relativity → Numerical Relativity

1. The mass integral formula: control the norm of the fields along the evolution. **It fixes the gauge**: maximal isothermal. It also suggest that **constrained evolution** schemes are better than free evolution schemes.
2. The role of extreme black holes as minimizer of the mass integral: suggests stability properties of these black holes. In particular, it suggests that it is perhaps easier to prove stability of an extreme black holes than a non-extreme one in axial symmetry.

Structure of the evolution equations (twist free case $\omega = 0$)

Metric

$$h = -\frac{\alpha^2}{\rho^2} e^{-\sigma} dt^2 + e^{\sigma+2q} \left((d\rho + \beta^\rho dt)^2 + (dz + \beta^z dt)^2 \right).$$

Five variables: σ, q, α, β^ρ, β^z.
Five equations: coupled non-linear hyperbolic-elliptic system.

- 1– hyperbolic equation for q (non-linear wave equation).
- 4– elliptic equations for σ, α, β^ρ, β^z (2-gauge conditions, 2-constraint equations)

Constrained evolution scheme: the constrains are solved at each step of the evolution.

- This evolution scheme (with some variants) were used in the numerical works mentioned before but without noticing this property of the mass. That is, this gauge has not only desirable analytical properties but it is also useful for numerical studies.
- Other gauge choices in axial symmetry have been used by Rinne 05 and Sorkin 10.
- **It is expected that the gauge exists for all t even when singularities are formed (since it is a maximal gauge).**

Evolution in the maximal-isothermal gauge

- The very basic question of well-posedness of the equations in this gauge is open.
- This question is rather subtle because of the singular behavior at the axis mentioned above.
- There are many possible evolution schemes (even if we restrict ourselves to constrained systems). It is very likely that few of them (or may be only one) are well-posed. If this is the case, the resolution of the well-posedness question will lead us to select (or even discover) the correct evolution scheme.
- The first step is to study the **linearization** of the equations around fixed solutions.

Why this problem is relevant?

- Being the local existence problem so complicated in this gauge one can wonder what can be said about the global behavior of the evolution, which is, of course, the ultimate goal.
- However, many of the main complications of this gauge are already present in the well-posedness problem because they are related to the local behavior of the fields at the symmetry axis.
- If one can sort out the difficulties for the linearized system in a satisfactory way there is a good chance that the mass integral formula can be used to control the global evolution in some way.

Relevance of the linear problem

The well-posedness of the linear equations are also relevant in themselves for the following two reasons:

- The mass formula at the linear level can in principle be used to say something about linear stability in axial symmetry of a background solution like a black hole.
- The well-posedness of the linear equations and the mass formula give insight on appropriate boundary conditions on a bounded domain. In particular, the mass formula allows us to calculate the gravitational waves that leave or enter a bounded domain.

Linear equations (flat background, no twist)

The unknowns are only two functions v and β, they satisfy the following equations

$$\ddot{v} = \Delta v - \frac{\partial_\rho v}{\rho} + \rho \partial_\rho \left(\frac{\beta}{\rho} \right) \quad \textbf{wave equation}$$

$$\Delta \beta = \frac{2}{\rho} \left(\Delta v - \frac{\partial_\rho v}{\rho} \right) \quad \textbf{elliptic equation}$$

We are using coordinates (t, ρ, z), with $\Delta v = \partial_\rho^2 v + \partial_z^2 v$ and a dot denotes time derivative.
With respect to the previous expression for the spacetime metric we have the relations $\beta^\rho = \beta$ and $q = \dot{v}$.

Boundary and initial conditions

Boundary conditions for the elliptic equation:

$$\beta|_{\rho=0} = 0 \text{ at the axis}, \quad \beta = O(r^{-1}) \text{ at infinity},$$

where $r = \sqrt{\rho^2 + z^2}$.

Initial conditions for the wave equation:

$$v|_{t=0} = f, \quad \dot{v}|_{t=0} = g.$$

At the axis we require that the initial conditions vanished, namely

$$f|_{\rho=0} = 0, \quad g|_{\rho=0} = 0.$$

Mass conservation for the linear equations

The total mass of the system is given

$$m = \frac{1}{16}\int_{\mathbb{R}^2_+}\left(4\frac{|\partial v|^2}{\rho^2} + (\Delta v)^2 + |\partial\sigma|^2\right)\rho\, d\rho dz.$$

Where σ is given in terms of v by

$$\Delta\sigma + \frac{\partial_\rho \sigma}{\rho} = -\Delta\dot{v}.$$

For a solution we have $\dot{m} = 0$.

Results: numerical

Strong numerical evidence that the system is well-posed and that its solutions have the expected behavior. (O. Ortiz, S. D. 09).

Quasi-local mass

$$m_\Omega = \int_\Omega \epsilon\rho \, d\rho dz,$$

and quasi-local mass conservation formula

$$\dot{m}_\Omega = \oint_{\partial\Omega} \epsilon^A n_A,$$

where n^A is the unit normal of $\partial\Omega$. The quantity $\epsilon^A n_A$ measure how much energy is leaving or entering the domain.

Outer boundary conditions

- To model an isolated system on a finite grid it is important to prescribe boundary conditions such that the gravitational radiation leaves the domain. In general, this is a very difficult problem since it is not even clear what we mean by gravitational radiation at a finite distance.
- In our gauge the mass formula allow us to compute gravitational radiation on a bounded domain.

The mass formula suggests a particular kind of boundary conditions such that

$$\dot{m}_\Omega \leq 0,$$

at least for the class of initial conditions studied.
We emphasize however that we have not been able to prove this analytically.

Results: mathematical

- **We solved this linearized system explicitly in terms of integral transformations in a remarkable simple form** (M. Reiris–S. D. 10).
- This integral transformation plays the role as the Fourier transform for constant coefficient equations
- This representation is well suited to obtain useful estimates to apply in the non-linear case.

Quasi-local geometrical inequalities in axial symmetry

The Christodoulou mass of a black hole is defined as follows

$$m_{bh} = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}.$$

where A is the area of the horizon and J the angular momentum.
For the Kerr black hole: $m = m_{bh}$.
This formula trivially satisfies the inequality

$$\sqrt{|J|} \leq m_{bh}.$$

If we accept m_{bh} as the correct formula for the quasi-local mass of an axially symmetric black hole, then this inequality provides the, rather trivial, quasi-local version of $\sqrt{|J|} \leq m$ mentioned before.

Is m_{bh} the quasi-local mass for a non-stationary black hole?

Heuristic argument

- Consider the evolution of m_{bh}. By the area theorem, we know that the horizon area A will increase.
- If we assume axial symmetry, then the angular momentum will be conserved at the quasi-local level. On physical grounds, one would expect that in this situation the quasi-local mass of the black hole should increase with the area, since there is no mechanism at the classical level to extract mass from the black hole (no Penrose process in axial symmetry).
- **Then, one would expect that both the area A and the quasi-local mass m_{bh} should monotonically increase with time.**

The time derivative of m_{bh} is given by

$$\dot{m}_{bh} = \frac{\dot{A}}{32\pi m_{bh}} \left(1 - \left(\frac{8\pi J}{A}\right)^2\right),$$

were we have used that the angular momentum J is conserved. By the area theorem, we have

$$\dot{A} \geq 0.$$

Then $\dot{m}_{bh} \geq 0$ if and only if the following inequality is satisfied

$$|J| \leq 8\pi A.$$

It is natural to conjecture that this inequality should be satisfied for any horizon in an axially symmetric asymptotically flat initial data. With equality only for extreme Kerr black hole.

If it is true, then we have a non trivial monotonic quantity (in addition to the black hole area) m_{bh}

$$\dot{m}_{bh} \geq 0.$$

Non-trivial control on the evolution.

Results on horizon area–angular momentum inequality for axisymmetric black holes

- It is proved for stationary black holes with surrounding matter. J. Hennig, M. Ansorg, C. Cederbaum. 08.
- The first variation of the area is zero and the second variation is positive on extreme Kerr for cylindrical initial data. This provides strong analytical evidences that the inequality is true for cylindrical initial data (non-stationary). And hence, it is true for the non trivial family of spinning Bowen-York initial data. (S. D. 10)
- At least near equilibrium (where the inequality is expected to be valid) it provides a reasonable definition of temperature for non-stationary axially symmetric black holes and an 'extremality criteria' in the spirit of I. Booth, S. Fairhurst 08.

Open problems (short term)

Evolution (hyperbolic flavor)

- Well-posedness for the maximal isothermal gauge. Is the Hamiltonian constraint always solvable? Does the gauge breaks down? (Mathematical)
- Implementation of the radiation boundary conditions for the full equations. How much they improve the evolution scheme? (Numerical)
- It is possible to prove that for such boundary conditions the energy is always decreasing? May be with some extra assumptions (Mathematical). Or, find counterexamples. (Numerical)
- Optimal choice of variables and evolution scheme. (Mathematical, Numerical)

Initial data (elliptic flavor)

- Prove the inequality $|J| \leq 8\pi A$ for initial data close to Kerr (M. E. Gabach, S. D, working on it). (Mathematical)
- Give strong evidences (or find a counter example) of this inequality far from known solutions. For example, for multiple black holes. (Numerical)