The role of special conformal Killing tensors in General Relativity

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Special conformal Killing tensor L

$$L_{ij|k} = \frac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right)$$

Special conformal Killing tensor (SCK): a conformal Killing tensor with *special* properties.

Conformal Killing tensor L of metric g: symmetric type $(0,2)$ tensor field if there is a 1-form α such that

$$L_{(ij|k)} = \alpha_{(i}g_{jk)},$$

$|$ denotes covariant derivative w.r.to Levi-Civita connection of g.

If α exact: L is called CK of *gradient* type.

Special conformal Killing tensor:
a conformal Killing tensor of gradient type (trace type).

$$L_{ij|k} = \frac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right).$$

Trace on $i, j \Rightarrow \alpha = \mathrm{d}\,(\mathrm{tr}\, L)$.

Special conformal Killing tensors: interesting properties:

- construct Killing tensors
- play a role in theory of separation of variables of differential equations (Hamilton-Jacobi)
- study of projectively equivalent metrics

construct Killing tensors

Killing tensors

construct Killing tensors

Killing vector and Killing tensor fields:

- Killing vector field generates symmetries (isometries) of g
- "metric is unchanged along direction of Killing vectors"
    - free particle won't feel any forces in that direction
    - component of its momentum along that direction conserved
- Killing tensors: natural generalisation of Killing vectors: symmetric tensor satisying

$$\nabla_{(\sigma}K_{\mu_1\dots\mu_n)} = 0$$

construct Killing tensors

Killing tensors are very important in GR ...

- Second rank KT correspond to constants of motion quadratic in momenta
- they play important role in complete solution of the problem of geodesic motion
- KT related to theory of separation of variables of Ham. Jac. eq.

... but solving Killing equations is not easy.

construct Killing tensors

Constructing Killing tensors from SCK's: Let L be a (non-singular) SCK. Then it's cofactor A

$$A_{ij}L^{j}_{\ell} = (\det L)g_{i\ell}; \qquad (1)$$

is a Killing tensor: take covariant derivative of (1)

$$A_{ij|k} = (\det L)\left(\overline{L}_{ij}\overline{L}_{k\ell} - \frac{1}{2}\overline{L}_{ik}\overline{L}_{j\ell} - \frac{1}{2}\overline{L}_{i\ell}\overline{L}_{jk}\right)\alpha^{\ell},$$

satisfies Killing equation $A_{(ij|k)} = 0$.

construct Killing tensors

Let L, K be SCK's, then the cofactor of $aL + bK$ is Killing, $\forall a, b$ (at least those for which $aL + bK$ is non-singular).

If $K = I$ and L has functionally independent eigenvectors, we generate from one SCK n independent Killing tensors, one of which is g, another of which is cofactor L.

Eigenvectors of $aL + bI$ are eigenvectors of $L \Rightarrow$ Killing tensors have same eigenvectors, are simultaneously diagonal with L.

construct Killing tensors

Remark: space of solutions of the SCK equations: finite dimensional vector space ($\Rightarrow$ limited n° of KT).

From integrability conditions of structural equations:

maximal dimension $\frac{1}{2}(n+1)\,(n+2)$ achieved $\Leftrightarrow$ space of constant curvature:

- Ricci symmetric $R_{ij} = R_{ji}$
- projectively flat $R^i_{ljk} = \frac{1}{n-1}\left[R_{lj}\delta^i_k - R_{lk}\delta^i_j\right]$.

latter equation: condition for constant curvature in (pseudo-)Riemannian case.

construct Killing tensors

Note:

If a manifold (M, g) admits two independent SCK L, K (no non-trivial common invariant subspaces), and
if L has functionally independent eigenfunctions,
then (M, g) is a space of constant curvature.

separation of variables

Related to Killing tensor problem:

Separation of variables of differential equations

separation of variables

Torsion of Nijenhuis tensor ($(1, 1)$-type tensor) vanishes: consequence: if eigenfunctions of L simple and functionally independent, $\exists$ orthonormal basis of $T_x M$ w.r.to which $L(x)$ is diagonal:

- at each point x of M, $L(x)$ is an endomorphism of tangent space $T_x M$;
- n functions u^i exist, such that $u^i(x)$ is eigenvalue of $L(x)$ and Jacobian $\partial u^i / \partial x^j$ everywhere non-singular;
- u^i may be taken as local (orthogonal) coordinates:

$$L = \sum u^i \frac{\partial}{\partial u^i} \otimes \mathrm{d}u^i;$$

- u^i are orthogonal separation coordinates for Hamilton-Jacobi equation for the geodesics of the manifold.

projective equivalence

Other application fields:

Projective equivalence

projective equivalence

g, h are proj. equiv. if they have same geodesics up to reparametrization.

$$L_{ij} = \left(\frac{\det h}{\det g}\right)^{1/(n+1)} g_{ik} g_{jl} h^{kl}$$

L is SCK of metric g, conversely given SCK L of g

$$h_{ij} = (\det L)^{-1} g_{ik} g_{jl} \overline{L}^{kl}$$

defines metric h projectively equivalent to g.

unfortunatly...

Problem: SCK equations are hard to solve in general

Aligned Petrov type D

Aligned SCK in Petrov type D space time : (SCK $\neq kg$)

- SCK only m, $\bar{m}$, k, l-contribution;
- SCK is a constant Killing tensor;
- $\Psi_i = 0$ except $\Psi_2 = -\frac{R}{12}$, R = Ricci constant;

  if R constant : Bertotti-Robinson;

- $\Phi_{ij} = 0$ except Φ_{11};

  if $\Phi_{11} = \pm\frac{R}{8}$: Plebanski-Hacyan, Garcia-Plebanski.

Thanks

Thank you for your attention!

structural equations

Tensorial quantities F^A are found, satisfying $F^A_{|i} = \Gamma^A_{Bi} F^B$, among which are equations of interest. F^A consist of tensor and tensors constructed from it and its covariant derivatives; coefficients Γ^A_{Bi} are tensorial quantities independent of F^A, built out of curvature and its covariant derivatives.

$$L_{ij|k} = \frac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right)$$

$$\alpha_{i|j} = \frac{1}{n}\left(2R^k_j L_{ik} - 2g^{kl} R^m_{ijk} L_{lm} + g_{ij}\mu\right)$$

$$\mu_{|i} = \frac{2}{n-1}\left(g^{jl}\left(2R^k_{i|l} - R^k_{l|i}\right) L_{jk} + (n+1)\,\alpha_j R^j_i\right)$$

common invariant subspaces

Consider type $(1,1)-$ tensors corresponding to K and L.
Subspace V of T_xM is common invariant subspace of K and L if $K(x)V \subset V$ and $L(x)V \subset V$.
Tensors K and L are independent if only common invariant subspaces at any point x are 0 and T_xM.

torsion

SCK is torsionless:

$$L^h_{[i}\nabla_{|h|}L^k_{j]} - L^k_l\nabla_{[i}L^l_{j]} = 0.$$