Membrane Paradigm and Holographic Hydrodynamics

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September 6, 2010

Holographic Principle

- Microscopic gravity dof in a volume V encoded on a boundary A of region

$$S_{BH} = \frac{A}{4\ell_p^2} \tag{1}$$

- Gravity (gravitational collapse) requires this is a maximal entropy: $S \leq S_{BH}$
- Realized the AdS/CFT correspondence
  - gravity in $(d+1)$ asymptotically AdS spacetimes dual to a CFT on the d-dimensional boundary
  - extra radial coordinate r of the AdS bulk $\rightarrow$ emergent dimension, coarse graining scale

Relativistic Hydrodynamics

- Universal description of large scale (long time, wavelength) dynamics of a field theory
- Regime where the *Knudsen* number $\frac{\ell_{corr}}{L} \ll 1$
- Equation of Motion: $\partial_\nu T^{\mu\nu} = 0$, for a CFT $\ell_{corr} \sim T^{-1}$, $T^\mu_\mu = 0$
- Constitutive relation

$$T^{\mu\nu} = \sum_{l=0} T_l^{\mu\nu}(x),\ T_l^{\mu\nu} \sim (Kn)^l \tag{2}$$

- l=0 ideal hydro, l=1 viscous, i.e.

$$T_{\mu\nu} \sim T^{d+1}[\eta_{\mu\nu} + (d+1)u_\mu u_\nu] - 2\eta\sigma_{\mu\nu} \tag{3}$$

  - $\sigma_{\mu\nu}$ shear tensor, η is the shear viscosity

Holographic Hydrodynamics

- AdS/CFT implies hydrodynamics of a CFT $\rightarrow$ large scale perturbations of AdS black hole solution (Bhattacharyya, Hubeny, Minwalla, and Rangamani, 2008)
- Consider a uniformly boosted black brane w/ temperature T and $u^{\mu} = (\beta, \beta v^{i})$
- Solve full set of field equations order by order in Knudsen number (derivatives ∂u, ∂T)
- GR constraint equations $G_{\mu B} n^{B} = 0$, $n_{A} = (dr)_{A}$ are the hydro equations. Notation: coordinates $X^{A} = (r, x^{\mu})$

Membrane Paradigm

- Classical GR- Any BH has *fictitious* viscous fluid living on its horizon (Damour 1979, 1982; Price, Thorne, et.al 1986)
- Can we work directly at the event horizon?
- Consider an equilibrium thermal state, i.e. a solution w/ timelike Killing vector, stationary horizon
- Expand set of Einstein's equations projected into the horizon $G_{\mu B}\ell^{B} = 0$ in Knudsen number (when permissible)

Horizon geometry

- Set horizon at $r = 0$
- Null normal: $\ell^A = g^{AB}\partial_B r$, set $g^{rr} = 0$ and $g^{r\mu} = \ell^\mu$ on horizon
- Intrinsic metric: $\gamma_{\mu\nu}$ pullback of g_{AB}. Degenerate- $\gamma_{\mu\nu}\ell^\nu = 0$
- Second fundamental form: $\theta_{\mu\nu} = \frac{1}{2}\mathcal{L}_\ell \gamma_{\mu\nu}$. Separate into shear and expansion

$$\theta_{\mu\nu} = \sigma^{(H)}_{\mu\nu} + \frac{1}{d-1}\gamma_{\mu\nu}\theta \tag{4}$$

- “Weingarten map" Θ^ν_μ

$$\Theta^\nu_\mu = \nabla_\mu \ell^\nu, \;\; \Theta^\rho_\mu \gamma_{\rho\nu} = \theta_{\mu\nu}, \;\; \Theta^\nu_\mu \ell^\mu = \kappa \ell^\nu \tag{5}$$

- Work with object $Q^\nu_\mu = v(\Theta^\nu_\mu - \kappa\delta^\nu_\mu)$, v horizon area density

- Intrinsic metric is degenerate, no unique intrinsic connection on a null surface
- Is a well-defined covariant divergence (Jezierski, Kijowski, and Czuchry, PRD 65 064036 (2002))

$$\bar{\nabla}_\nu Q^\nu_\mu = \partial_\nu Q^\nu_\mu - \frac{1}{2} Q^{\nu\rho} \partial_\mu \gamma_{\nu\rho} \tag{6}$$

- (Contracted) Gauss-Codazzi equations on a null surface

$$v R_{\mu\nu} \ell^\nu = \bar{\nabla}_\nu Q^\nu_\mu - v \partial_\mu \theta \tag{7}$$

- Can show the null focusing equation

$$R_{\mu\nu} \ell^\mu \ell^\nu = -\ell^\mu \partial_\mu \theta + \kappa\theta - \frac{1}{d-1}\theta^2 - \sigma^{(H)}_{\mu\nu} \sigma^{\mu\nu}_{(H)} \tag{8}$$

Black Branes in AdS

- Field equations

$$R_{AB} + dg_{AB} = 0 \tag{9}$$

- Boosted zeroth order solution

$$ds^2_{(0)} = -2\ell_\mu dx^\mu dr - (r+R)^2 f \ell_\mu \ell_\nu dx^\mu dx^\nu + (r+R)^2 (\eta_{\mu\nu} + \ell_\mu \ell_\nu) dx^\mu dx^\nu \tag{10}$$

$f = 1 - \frac{R^4}{r^4}$

- Can calculate the following quantities:

$$\gamma_{\mu\nu} = R^2(\eta_{\mu\nu} + \ell_\mu \ell_\nu) = R^2 P_{\mu\nu} \tag{11}$$

$$\kappa = 2R \tag{12}$$

$$\theta = \partial_\mu \ell^\mu + (d-1)\ell^\mu \partial_\mu R \tag{13}$$

$$\sigma^{(H)}_{\mu\nu} = R^2 \left(P^\alpha_\mu P^\beta_\nu \partial_{(\alpha} \ell_{\beta)} - \frac{1}{d-1} P_{\mu\nu} (\partial_\gamma \ell^\gamma) \right) \tag{14}$$

- At lowest order in derivatives the projected equations are the equations of an ideal conformal fluid w/ pressure $p = \frac{R^d}{4\pi}$
- Focusing equation $\theta = 0$ implies fluid entropy conservation

$$\partial_\mu (s\ell^\mu) = 0, \ \ s = \frac{R^{d-1}}{4} \tag{15}$$

- Now what about viscous order? In Bhattacharyya, et. al need the first order metric- solve for full metric, i.e. integrate in r with regular horizon BC

$$g_{AB} = g^{(0)}_{AB} + g^{(1)}_{AB}(\partial u, \partial T) \tag{16}$$

- However, we can make use of symmetry, variable fixing, and structure of constraint equations to work only w/ zeroth order solution

Conformal structure and Viscous hydrodynamics

- Rescaling freedom in solution

$$\tilde{r} = e^{-\phi} r,\ \tilde{\ell}^{\mu} = e^{-\phi} \ell^{\mu},\ \tilde{\eta}_{\mu\nu} = e^{2\phi} \eta_{\mu\nu},\ \tilde{R} = e^{-\phi} R \tag{17}$$

- Demand this as local symmetry holding for slowly varying $\phi(x^{\mu})$
- Intrinsic metric, Q^{ν}_{μ} are invariants. Conformal structure constrains form of corrections to these variables at first order
- Arrive at viscous hydrodynamics equations for a conformal fluid
- Focusing equation $\rightarrow$ fluid entropy balance law

$$\partial_{\mu} S^{\mu} = \partial_{\mu}(s\ell^{\mu}) = \frac{2\eta}{T} \sigma_{\mu\nu} \sigma^{\mu\nu},\ \eta = \frac{R^{d-1}}{16\pi} \tag{18}$$

Non-abelian hydrodynamics with anomalies

- Can be extended to non-abelian hydrodynamics of gauge theories w/ triangle anomalies, dual to Yang-Mills-Chern-Simons solutions
- New result for transport coefficients and holographic entropy current. In the anomaly case the entropy current has a novel term proportional to the vorticity (forthcoming paper)

$$S^\mu = s\left(u^\mu + \frac{s^{2/3} T^2 n}{2^{17/3} p^2} P^\mu_\nu \partial^\nu \frac{\mu}{T}\right) + \frac{s \beta^3 \mu^3}{3 \pi p} \omega^\mu \quad (19)$$

- Positivity of divergence guaranteed by classical second law
- $U(1)$ case agrees with previous results (Son and Surowka, 2009)

Summary and Conclusions

- A path to viscous holographic hydrodynamics working *locally* at an event horizon
- In some approximation, the Einstein constraint equations are Navier-Stokes equations. Can we learn something from this?
- A certain non-relativistic limit: split into space+time (ct, x^i), introduce scaling $\partial_t \sim c^{-2}, v^i \sim c^{-1}, T = T_0(1 + c^{-2}P)$
  - Equations reduce to ordinary incompressible Navier-Stokes

$$\partial_i v^i = 0,\ \partial_t v_i + v^j \partial_j v_i + \partial_i P = (4\pi T_0)^{-1} \nabla^2 v_i \tag{20}$$

- Verified on both sides of the duality: (Fouxon and Oz; Bhattacharyya, Minwalla and Wadia 2009)
- Can horizon geometry+dynamics provide insights into the nature of turbulent flows? A geometrization of turbulence?