Emission and null coordinates: geometrical properties and physical construction

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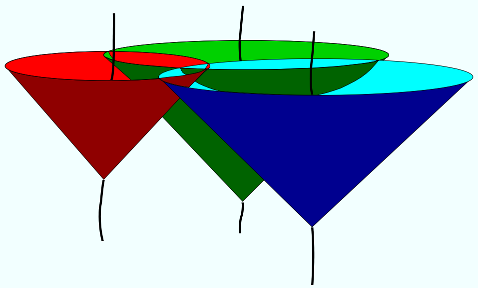


ERE-2010. Granada, September 2010

Relativistic positioning systems

Bartolomé Coll
Theory of Relativistic coordinate systems
ERE-2000, Valladolid

A Relativistic Positioning System is defined by four clocks γ_A (emitters) broadcasting their proper time τ^A

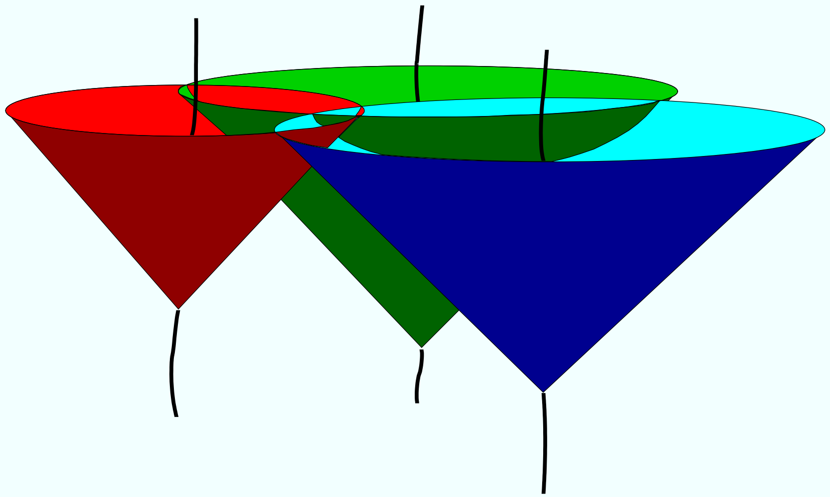
Relativistic Positioning Systems are physical realizations of emission coordinates

Emission coordinates

A Relativistic positioning system is defined by four clocks γ_A (emitters) broadcasting their proper time τ^A

The future light cones of the points $\gamma_A(\tau^A)$ constitute coordinate hypersurfaces $\tau^A = constant$ of a coordinate system.

At every event, four of these cones broadcasting the times τ^A intersect, endowing thus the event with the emission coordinates $\{\tau^A\}$: the four proper time signals received by any observer at the event from the four clocks.



(3-dimensional picture)

Bartolomé Coll and José M. Pozo
Relativistic positioning systems: the emission coordinates
Class. Quantum Grav. **23** (2006)

- The coordinate covectors of emission coordinates, $d\tau^A$, are null and future-directed.

- The contravariant metric in emission coordinates:

$$(g^{AB}) = \begin{pmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{pmatrix} \quad , \qquad g^{AB} < 0 \quad \text{for} \quad A \neq B$$

- The coordinate vectors $s_A \equiv \partial_{\tau^A}$ of emission coordinates are space-like.

Emission coordinates? Null coordinates?

- The coordinates associated with a Relativistic Positioning System have been considered by several authors. These coordinates receive different names in the literature: "null coordinates", "emission coordinates", "GPS coordinates", "GNSS coordinates". The first name, null coordinates, has been recommended enough because the name is a reference to their geometrical properties.

Emission coordinates? Null coordinates?

- ◇ The coordinates associated with a Relativistic Positioning System have been considered by several authors. These coordinates receive different names in the literature: "null coordinates", "emission coordinates", "GPS coordinates", "GNSS coordinates". The first name, null coordinates, has been recommended enough because the name is a reference to their geometrical properties.

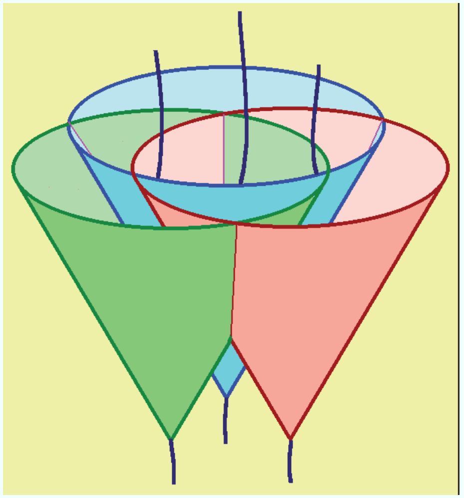
- ◇ Nevertheless, we (Bartolomé Coll and collaborators) prefer to call them emission coordinates.

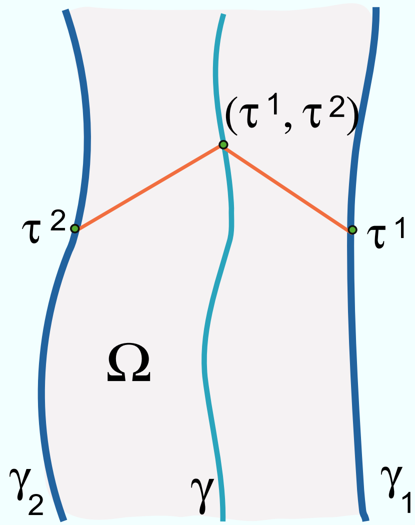
Why?

This talk

- Coordinates systems whose coordinate hypersurfaces are light cones based on world-lines:
  - Emission and reception coordinates
  - Emission-reception coordinates; radar coordinates
- Is the solar time a time-like coordinate?
- On the causal character of a coordinate:
  - Time-like, space-like or null gradient coordinate
  - Time-like, space-like or null coordinate parameter
- Some classes of null coordinates:
  - Null gradient coordinates
  - Null coordinate parameters
  - Bondi-Sachs coordinates

Emission coordinates



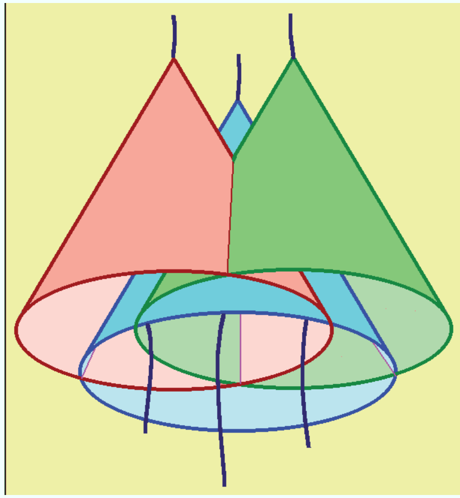


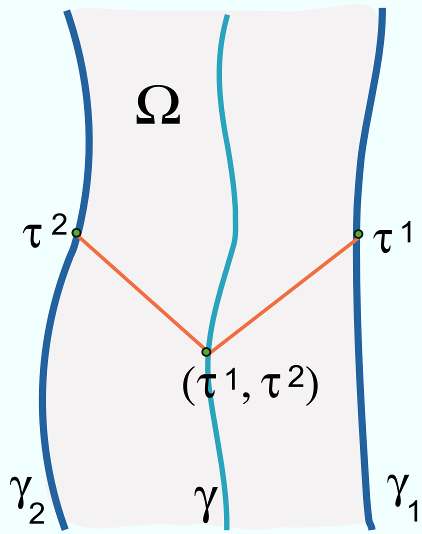
◇ The four families of coordinate hypersurfaces are future light cones with vertex on a world line (future-pointing time-like curve).

◇ The coordinate hypersurfaces are null: $g^{\alpha\alpha} = 0$.

Emission coordinates are null coordinates

Reception coordinates

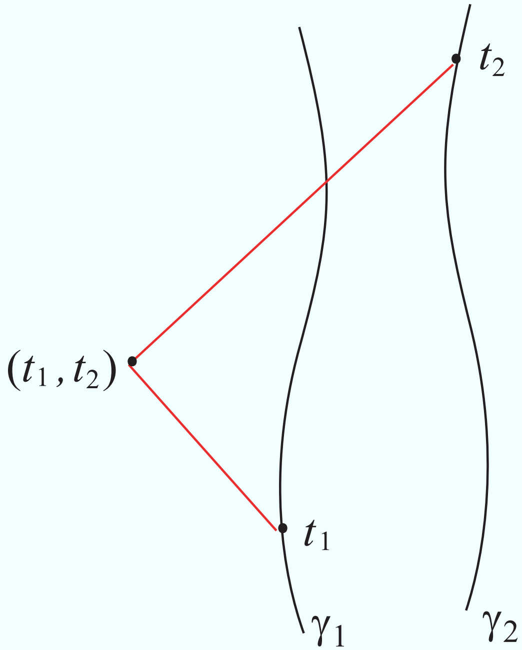




◇ The four families of coordinate hypersurfaces are past light cones with vertex on a world line (future-pointing time-like curve).

◇ The coordinate hypersurfaces are null: $g^{\alpha\alpha} = 0$.

Reception coordinates are null coordinates



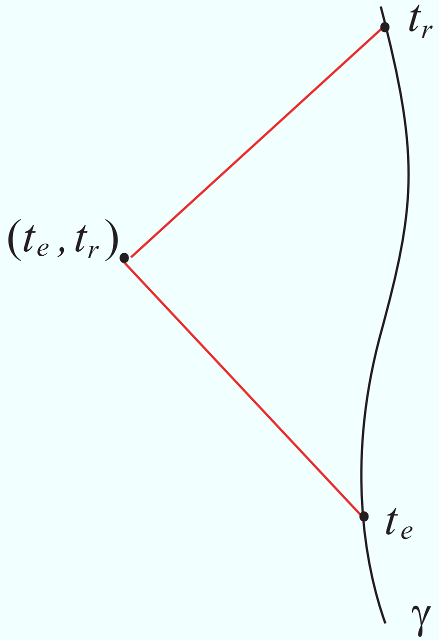
Emission-reception coordinates in two-dimensions:

an emitter γ_1 and a receiver γ_2

- ◇ The four families of coordinate hypersurfaces are future OR past light cones with vertex on a world line (future-pointing time-like curve).
- ◇ The coordinate hypersurfaces are null: $g^{\alpha\alpha} = 0$.

Emission-reception coordinates are null coordinates

Radar coordinates



- Radar coordinates $\{t_e, t_r\}$.
- Poincaré-Einstein coordinates $\{t, x\}$:

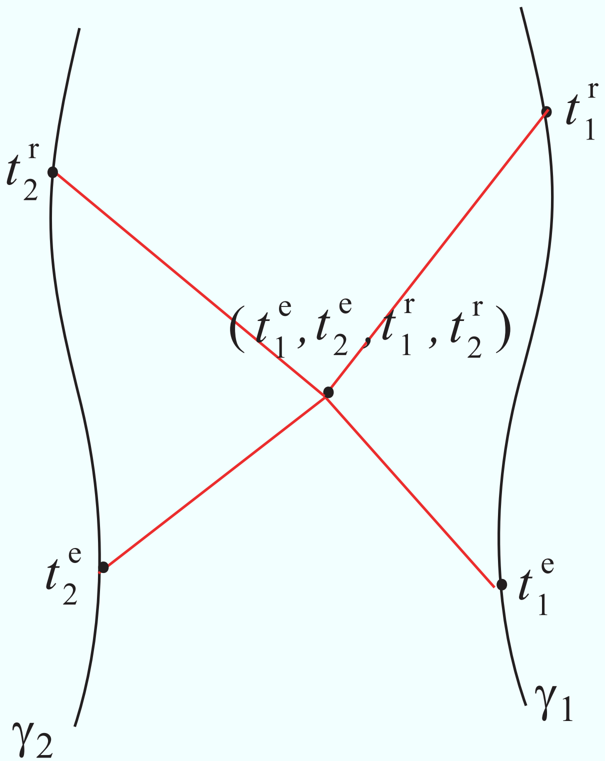
$$t = \tfrac{1}{2}(t_r + t_e)$$
$$x = \tfrac{1}{2}(t_r - t_e)$$

◇ Radar coordinates are based in the Poincaré-Einstein protocol of synchronization.

◇ Radar coordinates are emission-reception coordinates defined by a sole emitter-receiver.

Radar coordinates are null coordinates

Radar coordinates



Radar coordinates: $\{t_e^1,\ t_e^2,\ t_r^1,\ t_r^2\}$

J. Ehlers, F.A.E. Pirani, A.Schild (1972),
The geometry of free fall and light propagation

◇ Ehlers-Pirani-Schild radar coordinates are emission-reception coordinates defined by two emitter-receivers.

Ehlers-Pirani-Schild radar coordinates are null coordinates

In flat space-time:

- Inertial coordinates: $\{t,\, x,\, y,\, z\}$.
- Null coordinates ($g^{\alpha\alpha} = 0$): $\{u,\, v,\, w,\, s\}$,

$$u = t + x\,, \qquad v = t - x\,, \qquad w = t - y\,, \qquad s = t - z\,.$$

◇ They are null coordinates with coordinate hypersurfaces which are not light cones.

In flat space-time:

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- Null coordinates ($g^{\alpha\alpha} = 0$): $\{u,\, v,\, w,\, s\}$,

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◇ They are null coordinates with coordinate hypersurfaces which are not light cones.

In a generic space-time, four one-parametric families of null hypersurfaces define a coordinate system such that $g^{\alpha\alpha} = 0$

Null coordinates?

- Is the concept of null coordinates well understood?
- To associate a causal character to a coordinate, saying that it is time-like, space-like or null, is not generically coherent (Coll, Ferrando, Morales, Found. Phys. **39** (2009)).
- Thus, we must clarify what the denomination "null coordinate" means.

- Is the concept of null coordinates well understood?
- To associate a causal character to a coordinate, saying that it is time-like, space-like or null, is not generically coherent (Coll, Ferrando, Morales, Found. Phys. **39** (2009)).
- Thus, we must clarify what the denomination "null coordinate" means.

◇ Associated with a coordinate system $\{x^\alpha\}$ we have:

$$\text{frame} \quad \{\partial_\alpha\}\,, \qquad\qquad \text{co-frame} \quad \{dx^\alpha\}\,,$$

◇ For a coordinate x^α, the two natural variations in the coordinate system, dx^α and ∂_α, have generically different causal characters.

An example: the Solar time

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in Minkowski space-time:

- $\{t, \phi, \rho, z\}$: inertial cylindrical coordinate system.

$$\text{inertial time } t \begin{cases} \partial_t \quad \text{is time-like}: \ \text{t is a time-like coordinate parameter} \\ dt \quad \text{is time-like}: \ \text{t is a time-like gradient coordinate} \end{cases}$$

in Minkowski space-time:

- $\{t, \phi, \rho, z\}$: inertial cylindrical coordinate system.

$$\text{inertial time } t \begin{cases} \partial_t & \text{is time-like : t is a time-like coordinate parameter} \\ dt & \text{is time-like : t is a time-like gradient coordinate} \end{cases}$$

- $\{T, \Phi, \rho, z\}$: Solar time rotating cylindrical coordinate system.

$$T = \frac{\phi}{\omega}\,, \qquad \Phi = \phi - \omega t$$

In the interior $\rho < 1/\omega$ of the light cylinder:

$$\text{Solar time } T \begin{cases} \partial_T & \text{is time-like : T is a time-like coordinate parameter} \\ dT & \text{is space-like : T is a space-like gradient coordinate} \end{cases}$$

Gradient coordinates and coordinate parameters

Generically, we say (Coll, Ferrando, Morales, Found. Phys. **39** (2009)) that a coordinate x^α is a:

- time-like, space-like or null gradient coordinate when the causal character of its variation dx^α is time-like, space-like or null.
- time-like , space-like or null coordinate parameter when the causal character of its variation ∂_α is time-like, space-like or null.

Gradient coordinates and coordinate parameters

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- time-like , space-like or null coordinate parameter when the causal character of its variation ∂_α is time-like, space-like or null.

> What about null gradient coordinates and null coordinate parameters?

B. Coll, J.A. Morales, Int. Jour. Theor. Phys. **31** (1992)

199 Causal Classes of Space-Time Frames

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT		TTTEEE TTTLEE TTTTEE TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTTTE TTTTTL TTTTTT	TTLTLL TTTTLL TTTTTL TTTTTT	TTTTLL TTTTTL TTTTTT	TTTTTL TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT
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tlll	EEEEEE															
ttll	EEEEEE															
tttl	EEEEEE															
tttt	EEEEEE															

e, **E**, *e* space-like
t, **T**, *t* time-like
l, **L**, *l* light-like (null)

B. Coll, J.J. Ferrando, J.A. Morales, Phys. Rev. D **80** (2009)
Newtonian and Lorentzian emission coordinates

Null coordinates (parameters): $g_{\alpha\alpha} = 0$

- Causal signature (Coll, Morales, Int. J. Theor. Phys. **31** (1992)):

$$\boxed{\{\mathrm{l\,l\,l\,l},\ \mathrm{T\ T\ T\ T\ T\ T},\ e\ \ e\ \ e\ \ e\}}$$

  - Four null congruences of coordinate lines: $g_{\alpha\alpha} = 0$.
  - Six time-like families of coordinate surfaces.
  - Four time-like families of coordinate hypersurfaces.

- Physical construction by means of light beams

  B. Coll, *Light Coordinates in Relativity* ERE-1985; http//coll.cc

Null (gradient) coordinates: $g^{\alpha\alpha} = 0$

- Causal signature (Coll, Morales, Int. J. Theor. Phys. **31** (1992)):

$$\boxed{\{\mathrm{e\ e\ e\ e,\ E\ E\ E\ E\ E\ E},\ \ell\ \ell\ \ell\ \ell\}}$$

  - Four space-like congruences of coordinate lines.
  - Six space-like families of coordinate surfaces.
  - Four null families of coordinate hypersurfaces: $g^{\alpha\alpha} = 0$.

- Physical construction by emission and/or reception times: the coordinate hypersurfaces are future or past light cones.

  - Emission coordinates (relativistic GNSS systems).
    Coll, *Elements for a theory of relativistic coordinate systems. Formal and physical aspects*, ERE-2000; http//coll.cc
  - Reception coordinates (relativistic stereometry).
    Coll, *Epistemic relativity* (2008),
    http://www.uib.es/depart/dfs/GRG/GraviMAS_FEST/
  - Radar coordinates.

B. Coll, J.A. Morales, Int. Jour. Theor. Phys. **31** (1992)

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	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT		TTTEEE TTTLEE TTTTEE TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTTTE TTTTTL TTTTTT	TTLTLL TTTTLL TTTTTL TTTTTT	TTTTLL TTTTTL TTTTTT	TTTTTL TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT
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tlee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE TLEEEE	TLLEEE														
ttee	EEEEEE LEEEEE TEEEEE															
llle	EEEEEE LEEEEE LLEEEE LLELEE															
tlle	EEEEEE LEEEEE LLEEEE															
ttle	EEEEEE LEEEEE															
ttte	EEEEEE															
llll	EEEEEE															
tlll	EEEEEE															
ttll	EEEEEE															
tttl	EEEEEE															
tttt	EEEEEE															

Bondi-Sachs causal classes

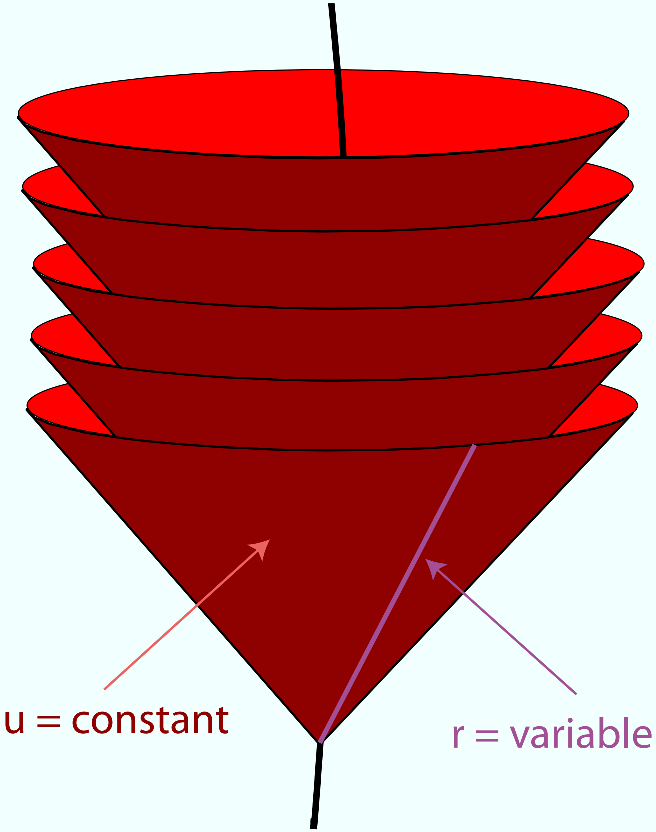
e, E, *e* space-like
t, T, *t* time-like
l, L, *l* light-like (null)

B. Coll, J.J. Ferrando, J.A. Morales, Phys. Rev. D **80** (2009)

Newtonian and Lorentzian emission coordinates

Bondi-Sachs coordinates: $g^{00} = 0, \; g_{11} = 0$

$$\{u,\, r,\, x^2,\, x^3\} \begin{cases} u \text{ is a null gradient coordinate}: & g^{00} = g(\mathrm{d}u, \mathrm{d}u) = 0 \\ r \text{ is a null coordinate parameter}: & g_{11} = g(\partial_r, \partial_r) = 0 \end{cases}$$



The orthogonal lines to the (null) coordinate hypersurfaces $u = constant$ are the (null) coordinate lines $r = variable$.

$$\boxed{\partial_r \propto g(\mathrm{d}u)}$$

Bondi-Sachs coordinates: $g^{00} = 0, \; g_{11} = 0$

- In flat space-time: starting from the spherical inertial coordinates $\{t, \, r, \, \theta, \, \phi\}$ we define the coordinate system $\{u, \, r, \, \theta, \, \phi\}, \;\; u = t \pm r$.
  - Flat metric: $ds^2 = -du^2 \pm 2\, du\, dr + r^2 d\Omega^2$.
  - u is a null gradient coordinate: $g^{00} = g(\mathrm{d}u, \mathrm{d}u) = 0$.
  - r is a null coordinate parameter: $g_{11} = g(\partial_r, \partial_r) = 0$.

- Schwarzschild space-time: starting from the Schwarzschild coordinates $\{t, \, r, \, \theta, \, \phi\}$, the Eddington-Finkelstein coordinates $\{u, \, r, \, \theta, \, \phi\}$ are given by the transformation $u = t \pm (r + 2m \ln |r - 2m|)$.
  - Schwarzschild solution: $ds^2 = \left(\frac{2m}{r} - 1\right) du^2 \pm 2\, du\, dr + r^2 d\Omega^2$.
  - u is a null gradient coordinate: $g^{00} = g(\mathrm{d}u, \mathrm{d}u) = 0$.
  - r is a null coordinate parameter: $g_{11} = g(\partial_r, \partial_r) = 0$.

Bondi-Sachs coordinates: $g^{00} = 0, \; g_{11} = 0$

- In flat space-time: starting from the spherical inertial coordinates $\{t,\, r,\, \theta,\, \phi\}$ we define the coordinate system $\{u,\, r,\, \theta,\, \phi\}$, $\; u = t \pm r$.
  - Flat metric: $\; ds^2 = -du^2 \pm 2\, du\, dr + r^2 d\Omega^2$.
  - u is a null gradient coordinate: $\; g^{00} = g(\mathrm{d}u, \mathrm{d}u) = 0$.
  - r is a null coordinate parameter: $\; g_{11} = g(\partial_r, \partial_r) = 0$.
- Vaidya space-time: is usually given in Eddington-Finkelstein-like coordinates $\{u,\, r,\, \theta,\, \phi\}$.
  - Vaidya radiating solution: $\; ds^2 = \left(\frac{2m(r)}{r} - 1\right) du^2 - 2\, du\, dr + r^2 d\Omega^2$.
  - u is a null gradient coordinate: $\; g^{00} = g(\mathrm{d}u, \mathrm{d}u) = 0$.
  - r is a null coordinate parameter: $\; g_{11} = g(\partial_r, \partial_r) = 0$.

B. Coll, J.A. Morales, Int. Jour. Theor. Phys. **31** (1992)

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leee	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEE TLEEEE TEELEE EETLEE TTEEEE LLLEEE LLELEE LELLEE TLLEEE TLELEE TELLEE LETLEE TTLEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TTLLEE TLTLEE TTTLEE	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE	TEELLE TLELLE TTELLE TLLLLE TTLLLE TTTLLE	TTTEEE TTTLEE	TTLLLE TTTLLE	TTTLLE										
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tttl	EEEEEE															
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Bondi-Sachs causal classes

Causal classes of Eddington-Finkelstein coordinates

e, E, *e*	space-like
t, T, *t*	time-like
l, L, *l*	light-like (null)

B. Coll, J.J. Ferrando, J.A. Morales, Phys. Rev. D **80** (2009)
Newtonian and Lorentzian emission coordinates