A special class of Petrov type D vacuum space-times in dimension five.

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The role of the G.H.P. formalism in General Relativity

The Geroch, Held, Penrose (G.H.P.) formalism takes as a structure group the group of boosts and spins of a given null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$.

$$z \in \mathbb{C}\,, \quad l^a \mapsto z\bar{z} l^a\,, \quad n^a \mapsto \frac{n^a}{z\bar{z}}\,, \quad \text{Boost.}$$

$$m^a \mapsto \frac{z}{\bar{z}} m^a\,, \quad \bar{m}^a \mapsto \frac{\bar{z}}{z} \bar{m}^a\,, \quad \text{Spin.}$$

Consider only NP quantities which behave well under boost and spin transformations (weighted quantities).

$$Q \mapsto z^p \bar{z}^q Q,$$

The quantity Q has weight (p, q) (boost weight $(p+q)/2$, spin weight $(p-q)/2$).

The role of the G.H.P. formalism in General Relativity

Some references:

- "A space-time calculus based on pairs of null directions", R. Geroch, A. Held and R. Penrose (1973).
- "Integration in the GHP formalism I-IV" S. B. Edgar and G. Ludwig. (1996-2000).
- "The Karlhede classification of type D vacuum spacetimes" J. M. Collins, R. A. d'Inverno and J. A. Vickers (1990).
- "Petrov D vacuum spaces revisited: identities and invariant classification", S. B. Edgar, A. García-Parrado and J. M. Martín-García (2009).

A generalisation of the Newmann-Penrose formalism to dimension five

We discuss next a generalisation of the Newman-Penrose formalism to dimension five (signature convention $(-,+,+,+,+)$).

Semi null pentad:

$$N \equiv \{l^a, n^a, m^a, \bar{m}^a, u^a\}, \quad l^a n_a = -1\ , \quad m^a \bar{m}_a = 1\ , \quad u^a u_a = 1.$$

Semi-null pentad frame derivations:

$$D \equiv l^a \nabla_a\ , \quad \Delta \equiv n^a \nabla_a\ , \quad \delta \equiv m^a \nabla_a\ , \quad \bar{\delta} \equiv \bar{m}^a \nabla_a\ , \quad \mathcal{D} \equiv u^a \nabla_a.$$

The spin coefficients

The spin coefficients

$$\nabla_b l_a = \mathfrak{e} l_b u_a + \mathfrak{d} n_b u_a - \mathfrak{f} u_a u_b - l_a l_b(\gamma + \bar{\gamma}) - l_a n_b(\epsilon + \bar{\epsilon}) + (\theta + \bar{\theta}) l_a u_b +$$
$$+2\mathrm{Re}\,(-m_b \bar{m}_a \rho - \bar{m}_a \bar{m}_b \sigma - \bar{m}_b u_a \varsigma + l_b \bar{m}_a \tau + \bar{m}_a n_b \kappa + \bar{m}_a u_b \eta + l_a \bar{m}_b(\bar{\alpha} + \beta))$$
$$\nabla_b n_a = -\mathfrak{b} l_b u_a - \mathfrak{a} n_b u_a + \mathfrak{c} u_a u_b + l_b n_a(\gamma + \bar{\gamma}) + n_a n_b(\epsilon + \bar{\epsilon}) - n_a u_b(\theta + \bar{\theta}) +$$
$$+2\mathrm{Re}\,(-l_b m_a \nu + \bar{m}_b u_a \xi - m_a n_b \pi - \bar{m}_b n_a(\bar{\alpha} + \beta) + m_a u_b \zeta + m_a m_b \lambda + m_a \bar{m}_b \mu)$$
$$\nabla_b m_a = m_a\Big((\theta - \bar{\theta}) u_b + (\beta - \bar{\alpha}) \bar{m}_b + (\alpha - \bar{\beta}) m_b + (-\gamma + \bar{\gamma}) l_b + (-\epsilon + \bar{\epsilon}) n_b\Big) +$$
$$+u_b(l_a \bar{\zeta} + n_a \eta) - l_a(\bar{\lambda} \bar{m}_b + \bar{\mu} m_b - \bar{\nu} l_b) - \bar{\pi} l_a n_b + n_a(\kappa n_b + \rho m_b - \sigma \bar{m}_b + \tau l_b) -$$
$$-u_a(\upsilon m_b + \phi \bar{m}_b) + \chi n_b u_a - \psi u_a u_b + \omega l_b u_a\ ,$$
$$\nabla_b u_a = -\mathfrak{b} l_a l_b + \mathfrak{e} l_b n_a - \mathfrak{a} l_a n_b + \mathfrak{d} n_a n_b + \ \mathfrak{c} l_a u_b - \mathfrak{f} n_a u_b +$$
$$2\mathrm{Re}\Big(- \bar{m}_b n_a \varsigma + l_a \bar{m}_b \xi + m_b \bar{m}_a \upsilon + \bar{m}_a \bar{m}_b \phi - \bar{m}_a n_b \chi + \bar{m}_a u_b \psi - l_b \bar{m}_a \omega\Big).$$

The spin coefficients

Twelve Newman-Penrose 4-D spin coefficients

α, β, γ, ϵ, κ, λ, μ, ν, π, ρ, σ, τ.

Ten complex 5-D spin coefficients

ζ, η, θ, χ, ω, ϕ, ξ, υ, ψ, ς.

Six real 5-D spin coefficients

$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{f}$.

$2 \times 12 + 2 \times 10 + 6 = 50$ real Ricci rotation coefficients.

Weyl scalars

$$l^a n^b l^c n^d W_{abcd} = -(\Psi_2 + \overline{\Psi}_2) + 2\Psi_{11}\ ,\ l^a n^b l^c m^d W_{abcd} = \Psi_1 - \Psi_{01}\ ,$$
$$l^a n^b l^c u^d W_{abcd} = \ {}^*\Psi_1 + \ {}^*\overline{\Psi}_1\ ,\ l^a n^b n^c m^d W_{abcd} = \Psi_{12} - \overline{\Psi}_3\ ,$$
$$l^a n^b n^c u^d W_{abcd} = \Psi_3^* + \overline{\Psi}_3^*\ ,\ l^a n^b m^c \bar{m}^d W_{abcd} = \Psi_2 - \overline{\Psi}_2\ ,$$
$$l^a n^b m^c u^d W_{abcd} = -\ {}^*\overline{\Psi}_2 - \Psi_2^*\ ,\ l^a m^b l^c m^d W_{abcd} = -\Psi_0\ ,\ l^a m^b l^c \bar{m}^d W_{abcd} = \Psi_{00}\ ,$$
$$l^a m^b n^c m^d W_{abcd} = -\Psi_{02}\ ,\ l^a m^b n^c \bar{m}^d W_{abcd} = \Psi_2\ ,\ l^a m^b n^c u^d W_{abcd} = -\Psi_2^*\ ,$$
$$l^a m^b m^c \bar{m}^d W_{abcd} = -\Psi_1 - \Psi_{01}\ ,\ l^a m^b m^c u^d W_{abcd} = \Psi_1^*\ ,\ l^a m^b \bar{m}^c u^d W_{abcd} = \ {}^*\Psi_1\ ,$$
$$l^a u^b l^c u^d W_{abcd} = -2\Psi_{00}\ ,\ l^a u^b n^c m^d W_{abcd} = \ {}^*\overline{\Psi}_2\ ,$$
$$l^a u^b m^c \bar{m}^d W_{abcd} = -\ {}^*\Psi_1 + \ {}^*\overline{\Psi}_1\ ,\ l^a u^b m^c u^d W_{abcd} = 2\Psi_{01}\ ,\ n^a m^b n^c m^d W_{abcd} = -\overline{\Psi}_4\ ,$$
$$n^a m^b n^c \bar{m}^d W_{abcd} = \Psi_{22}\ ,\ n^a m^b n^c u^d W_{abcd} = \overline{\Psi}_4^*\ ,$$
$$n^a m^b m^c u^d W_{abcd} = -\ {}^*\overline{\Psi}_3\ ,\ n^a m^b \bar{m}^c u^d W_{abcd} = -\overline{\Psi}_3^*\ ,\ n^a u^b n^c u^d W_{abcd} = -2\Psi_{22}\ ,$$
$$n^a u^b m^c \bar{m}^d W_{abcd} = -\Psi_3^* + \overline{\Psi}_3^*\ ,\ n^a u^b m^c u^d W_{abcd} = 2\Psi_{12}\ ,$$
$$m^a \bar{m}^b m^c \bar{m}^d W_{abcd} = -(\Psi_2 + \bar{\Psi}_2) - 2\Psi_{11}\ ,\ m^a \bar{m}^b m^c u^d W_{abcd} = \Psi_2^* - \ {}^*\overline{\Psi}_2\ ,$$
$$m^a u^b m^c u^d W_{abcd} = -2\Psi_{02}\ ,\ m^a u^b \bar{m}^c u^d W_{abcd} = -2\Psi_{11}\ ,\ l^a m^b l^c u^d W_{abcd} = -\ {}^*\Psi_0\ ,$$
$$l^a u^b n^c u^d W_{abcd} = -2\Psi_{11}\ ,\ n^a m^b m^c \bar{m}^d W_{abcd} = -\Psi_{12} - \overline{\Psi}_3\ ,$$

Weyl scalars

Five 4-D Newman-Penrose components

Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4.

Eleven 5-D complex components

$$^*\Psi_0\ ,\quad ^*\Psi_1\ ,\quad \Psi_1^*\ ,\quad ^*\Psi_2\ ,\quad \Psi_2^*\ ,\quad ^*\Psi_3\ ,\quad \Psi_3^*\ ,\quad \Psi_4^*\ ,\quad \Psi_{01}\ ,\quad \Psi_{02}\ ,\quad \Psi_{12}.$$

Three 5-D real components

Ψ_{00}, Ψ_{11}, Ψ_{22}.

2×(16 complex components)+3 real components=35.

Ricci scalars

S_{ab} is the trace-free part of the Ricci tensor.

$$
\begin{aligned}
&l^a l^b S_{ab} = 3\Phi_{00}\ , l^a n^b S_{ab} = 3\Phi_{11}\ , l^a m^b S_{ab} = -3\Phi_{01}\ , \\
&l^a S_{ab} u^b = -3\ {}^*\Phi_{01}\ , n^a n^b S_{ab} = 3\Phi_{22}\ , m^b n^a S_{ab} = -3\Phi_{12}\ , \\
&n^a S_{ab} u^b = -3\ {}^*\Phi_{12}\ , m^a m^b S_{ab} = 3\Phi_{02}\ , \\
&m^a \bar{m}^b S_{ab} = 3\Phi_{11} - 3\Omega,\ m^a S_{ab} u^b = 3\ {}^*\Phi_{02}\ , S_{ab} u^a u^b = 6\Omega
\end{aligned}
$$

4-D Newman-Penrose components

Real: Φ_{00}, Φ_{11}, Φ_{22}, Complex: Φ_{01}, Φ_{02}, Φ_{12}.

5-D components

Real: Ω, ${}^*\Phi_{01}$, ${}^*\Phi_{12}$, Complex: ${}^*\Phi_{02}$.

2×(4 complex components)+6 real components=14.
We define the scalar curvature as $\Lambda = -R/20$.

The extension of the G.H.P. formalism to dimension five

One can carry out the ideas of the G.H.P. formalism to dimensions higher than four.

The G.H.P. formalism in dimension d (Durkee, Pravda, Pravdová and Reall, 2010)

Define the tetrad $\{\vec{\boldsymbol{l}}, \vec{\boldsymbol{n}}, \vec{\boldsymbol{m}}_{(i)}\}$, $i = 2, \cdots, d-1$. One introduces next boosts and spins with respect to this tetrad

$$\vec{\boldsymbol{l}} \mapsto \lambda \vec{\boldsymbol{l}}, \quad \vec{\boldsymbol{n}} \mapsto \frac{1}{\lambda} \vec{\boldsymbol{l}}, \quad \lambda \in \mathbb{R}, \quad \text{Boost}$$

$$\vec{\boldsymbol{m}}_{(i)} \mapsto X_{ij} \vec{\boldsymbol{m}}_{(j)}, \quad X_{ij} \in SO(d-2), \quad \text{Spin}$$

The G.H.P. formalism in dimension $d = 5$

We consider a set-up in which there is an invariant spacelike vector $\vec{\boldsymbol{u}}$. Introduce the semi-null pentad N and use boosts and spins of the four-dimensional part to construct the G.H.P. formalism.

The extension of the G.H.P. formalism to dimension five

Weighted G.H.P. operators

$$\text{þ}Q \equiv (D - p\epsilon - q\bar{\epsilon})Q \ , \ \text{þ}'Q \equiv (\Delta - p\gamma - q\bar{\gamma})Q \ ,$$
$$\eth' Q \equiv (\bar{\delta} - p\alpha - q\bar{\beta})Q \ , \ \widehat{\mathcal{D}}Q \equiv (\mathcal{D} - p\theta - q\bar{\theta})Q \ ,$$
$$\eth Q \equiv (\delta - p\beta - q\bar{\alpha})Q \ ,$$

We show the weighted spin coefficients toghether with their weights

$$\mathfrak{a} : (0,0) \ , \ \mathfrak{b} : (-2,-2) \ , \ \mathfrak{c} : (-1,-1) \ , \ \mathfrak{d} : (2,2) \ , \ \mathfrak{e} : (0,0) \ , \ \mathfrak{f} : (1,1) \ ,$$
$$\zeta : (-2,0) \ , \ \eta : (2,0) \ , \ \kappa : (3,1) \ , \ \lambda : (-3,1) \ , \ \mu : (-1,-1) \ , \ \nu : (-3,-1) \ ,$$
$$\xi : (0,-2) \ , \ \rho : (1,1) \ , \ \sigma : (3,-1) \ , \ \varsigma : (2,0) \ , \ \tau : (1,-1) \ , \ \upsilon : (0,0) \ ,$$
$$\phi : (2,-2) \ , \ \chi : (2,0) \ , \ \psi : (1,-1) \ , \ \omega : (0,-2) \ , \ \pi : (-1,1) \ ,$$

One can now compute the commutators of the Weighted operators, the Ricci identities and the Bianchi identities.

Petrov types in dimensions higher than four

It is possible to generalise the Petrov classification of the Weyl tensor to any dimension by means of the alignment theory (Milson, Coley, Pravda, Pravdová, 2005). One may define a number of generic Petrov types in any dimension. These are characterised by the Weyl aligned null directions (WANDS) and their alignment order.

Weyl tensor Petrov types for $d \geq 4$.

Petrov type	G	I	I_i	II	II_i	D	III	III_i	N	O
Alignment type	G	(1)	(1,1)	(2)	(2,1)	(2,2)	(3)	(3,1)	(4)	(5)

The Petrov type D in dimension five

In $d = 5$, when the Petrov type is D, there exists a semi-null pentad whose null elements are the aligned WANDS and such that the only nonvanishing components of the Weyl tensor are the following ones (zero boost Weyl scalars)

$$\Psi_{11}\ , \quad \Psi_2\ , \quad \Psi_{02}\ , \quad \Psi_2^*\ , \quad {}^*\Psi_2.$$

In this work we restrict ourselves to the case in which only $(0, 0)$ weighted Weyl scalars are not zero (the $\mathcal{A}$ class). This leaves us with

$$\Psi_{11}\ , \quad \Psi_2.$$

We will classify next all the possible vacuum solutions with cosmological constant Λ non-zero such that the only nonvanishing Weyl scalars are Ψ_{11} and Ψ_2.

The invariant classification of the $\mathcal{A}$ class

We summarise next the subcases included in the $\mathcal{A}$ class and their features. These results were derived using our extension of the G.H.P. formalism.

	$\Psi_{11} = 0$	$\bar{\Psi}_2 = \Psi_2$ $\Psi_2\bar{\Psi}_2 \neq 4\Psi_{11}^2 \neq 0$	$\Psi_2\bar{\Psi}_2 = 4\Psi_{11}^2$ $\bar{\Psi}_2 \neq \Psi_2$	$\Psi_2 = 2\Psi_{11}$	$\Psi_2 = -2\Psi_{11}$
Karlhede bound	**2**	≥ 1	≥ 1	≥ 1	≥ 1
Number of WANDS	**2**	**2**	**2**	∞	**2**
Global quantities (integration constants)	≤ 4	**2**	**3**	**1**	**1**
Number of linearly independent Killing Vectors	≥ 2	**4**	**3**	?	?

Remarks:

- Metrics in case $\Psi_2 = -2\Psi_{11}$ contain undetermined functions. Metrics in the remaining cases only depend on integration constants.

Further research

- Investigate other Petrov type vacuum solutions in 5 dimensions different from the class $\mathcal{A}$. The exhaustive classification of all the vacuum type D solutions in $d = 5$ seems to be very difficult. One should look for interesting subcases in which an exhaustive classification can be achieved.
- Explicit integration of the equations for the solutions belonging to the class $\mathcal{A}$.
- Our extended G.H.P. formalism could find applications in other contexts in which a space-like direction is naturally singled out. For example it could be used to study type D perfect fluid solutions in dimension five.

Computations done with *xAct* (www.xact.es) & Maple.