Classical general relativity as BF-Plebanski theory with linear constraints

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Motivation

In three dimensions, general relativity is topological.

$$S = \frac{1}{16\pi G}\int_\Sigma \epsilon_{ABC}E^A \wedge R^{BC}[\omega]$$

Quantisation of such a theory is well understood.
Idea: Try to write general relativity in 4D as a **constrained** topological theory

$$S = \int_\Sigma B^{AB} \wedge R_{AB}[\omega] + \lambda^\alpha C_\alpha[B];$$

the constraints should enforce $B^{AB} = \frac{1}{16\pi G}\epsilon^{AB}{}_{CD}E^C \wedge E^D$ to recover GR.

Addition of a Holst term is straightforward; then need $\Sigma^{AB} = \frac{1}{8\pi\gamma G}E^A \wedge E^B$, where $\Sigma^{AB} = \frac{1}{1\pm\gamma^2}\left(B^{AB} - \frac{\gamma}{2}\epsilon^{AB}{}_{CD}B^{CD}\right)$.

Motivation (II)

Traditional (Plebanski) formulation: Use **quadratic constraints**

$$\epsilon_{ABCD}\Sigma^{AB}_{ab}\Sigma^{CD}_{cd} = V\epsilon_{abcd}$$

These constraints have two **separate** sectors of solutions[1]

$$\text{either } \Sigma^{AB} = \pm e^A \wedge e^B \quad \text{or } \Sigma^{AB} = \pm\frac{1}{2}\epsilon^{AB}{}_{CD}e^C \wedge e^D$$

for some set of 1-forms e^A. Classically, one can consistently remain within the "GR" sector; quantum mechanically, the situation is less clear.

[1] under the additional assumption that $V \neq 0$! $V = 0$ configurations are not geometric at all.

Motivation (III)

This construction is used in discrete (spin foam) models of quantum gravity.
One introduces a triangulation of spacetime and integrates Σ^{AB} over triangles

$$\Sigma^{AB}_{ab}(x) \quad \Rightarrow \quad \Sigma^{AB}_{\triangle} \equiv \int_{\triangle} \Sigma^{AB} \in \mathfrak{so}(4) \simeq \Lambda^2\mathbb{R}^4$$

One then imposes the constraints

$$\epsilon_{ABCD}\Sigma^{AB}_{\triangle}\Sigma^{CD}_{\triangle'} = 0$$

if $\triangle = \triangle'$ or $\triangle$ and $\triangle'$ share an edge; the remaining constraints can be replaced by the "closure constraint"

$$\sum_{\triangle \subset ⟁} \Sigma^{AB}_{\triangle} = 0.$$

These constraints lead to the Barrett-Crane model (Barrett/Crane 1997).

Motivation (IV)

Recently, new spin foam models (EPR(L)/FK) have been proposed; these rely on the replacement of quadratic constraints on $\Sigma^{AB}_{\triangle}$ by **linear** constraints:

$$n_A(\text{\tetrahedron})\Sigma^{AB}(\triangle) = 0 \quad \forall\, \triangle \subset \text{\tetrahedron},$$

where $n_A(\text{\tetrahedron})$ is the normal to the tetrahedron $\text{\tetrahedron}$.

These are stronger than the quadratic constraints; they restrict Σ^{AB} to the discrete analog of

$$\Sigma^{AB} = \pm e^A \wedge e^B.$$

Our aim is to extend this construction to the **classical continuum theory**.

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Continuum construction

We need to introduce a basis of 1-forms e_a^A (i.e. we assume $\det(e_a^A) \neq 0$). Equivalently, we have a basis of 3-forms

$$n_{Adef} \equiv \epsilon_{ADEF} e_d^D e_e^E e_f^F, \quad e_A^c \sim \epsilon^{cdef} n_{Adef}$$

Claim 1. For a basis of 3-forms n_A, the general solution to

$$n_{Adef} \Sigma_{ab}^{AB} = 0 \quad \forall \{a,b\} \subset \{d,e,f\}$$

is

$$\Sigma_{ab}^{AB} = G_{ab} e_a^{[A} e_b^{B]},$$

where e^A is defined in terms of n_A as above. Note that $G_{ab} = G_{ab}(x)$.

Continuum construction (II)

One could try a linear redefinition $e_a^A = \lambda_a E_a^A$ to identify this general solution

$$\Sigma_{ab}^{AB} = G_{ab} e_a^{[A} e_b^{B]}$$

with $\Sigma^{AB} = \pm E^A \wedge E^B$, but this is not possible in general; one needs additional conditions.

Imposing the additional three constraints

$$\sum_b \sum_{\{a,f\} \not\in \{b,e\}} n_{Abef} \Sigma_{ab}^{AB} = 0, \quad e \in \{0,1,2\} \text{ fixed.}$$

implies that $G_{ab}(x) = c(x)$; can absorb this by pointwise rescaling[2] $E^A = \sqrt{|c|} e^A$.

[2] Note that $c(x) = 0$ at some points is not excluded, which will lead to $E^A = 0$ at these points.

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Discrete construction

The translation of the constraints into the discretised variables is straightforward. Introduce a triangulation, integrate Σ^{AB} over triangles and n_A over tetrahedra

$$\Sigma^{AB}_{ab},\ n_{Adef}\ \Rightarrow\ \Sigma^{AB}(\triangle),\ n_A(⟁)$$

The discrete analogue of

$$n_{Adef}\Sigma^{AB}_{ab}=0\quad\forall\{a,b\}\subset\{d,e,f\}$$

is (essentially by construction) the set of linear constraints used in EPR(L)/FK

$$n_A(⟁)\Sigma^{AB}(\triangle)=0\quad\forall\,\triangle\subset ⟁$$

Discrete construction (II)

More interestingly, the remaining constraints would take the form

$$\sum_{\{i,j\}\not\ni\mathbf{A}} n_A(\text{◬}_{\mathbf{i}})\Sigma^{AB}(\triangle_{\mathbf{Aj}}) = 0$$

Here, we label the tetrahedra in a 4-simplex by $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$; there are five of these constraints per simplex (where $\mathbf{A}$ is replaced by $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ respectively).

Result: These constraints follow from the EPR(L)/FK linear constraints, the closure constraint on $\Sigma^{AB}(\triangle)$, plus an analogous "4D closure constraint"

$$n_A(\text{◬}_{\mathbf{A}}) + n_A(\text{◬}_{\mathbf{B}}) + n_A(\text{◬}_{\mathbf{C}}) + n_A(\text{◬}_{\mathbf{D}}) + n_A(\text{◬}_{\mathbf{E}}) = 0$$

which can be given a clear geometric motivation, just as closure on Σ^{AB} !

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Summary

- Introducing a basis of 3-forms at each point, one can give a formulation of classical GR as BF theory plus linear constraints.

- The discrete version of the same action leads to variables $\Sigma^{AB}(\triangle)$, $n_A(\text{tetrahedron})$, which have to be constrained by the EPR(L)/FK linear constraints, plus a closure constraint on both Σ^{AB} and n_A, to reproduce the discrete analog of the continuum constraints.

- The 4D closure constraint suggests a new formulation in which the normals n_A are given a fully geometric role.

- Outlook: Canonical analysis of the continuum action; implications for spin foam models; relation to new GFT constructions; relation of our constraints to "edge simplicity" (Dittrich/Ryan), . . .

Thank you!