Generating Method Based on Conformal Invariance of the Maxwell Field

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Outline

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Generating Methods in General

- Exact solutions are difficult to find
- Generating method:
    1. Take an existing solution (*seed spacetime*)
    2. Apply some transformation
    3. Obtain a new solution (*new spacetime*)
- Method we use: conformal transformation

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Conformal Transformation

1. Seed spacetime: $g_{\mu\nu}$
2. Multiply metric by suitable scalar function Ω^2
3. New spacetime: $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

Note: Maxwell field is conformally invariant!

- If $F_{\mu\nu}$ is a solution of source-free Maxwell equations on background $g_{\mu\nu}$, it is also a solution on $\tilde{g}_{\mu\nu}$

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Theorems Concerning Conformal Transformation

Brinkmann

The only properly conformally related Einstein spaces ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) are vacuum pp-waves or Minkowsky and (A)dS.

Daftardar-Gejji

If $\tilde{G}_{\mu\nu} = G_{\mu\nu}$ then both spacetimes are pp-waves.

- Many other theorems suggesting that *vacuum* is not very suitable seed spacetime...

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1. Motivation
   - Generating Technique
   - Existing Theorems
2. Our Generating Method
   - Introducing the Method
   - Application

Main Idea

1. Seed spacetime: Solution of Einstein-Maxwell equations $(g_{\mu\nu}, F_{\mu\nu})$
2. Conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$
3. New spacetime: Solution of Einstein-Maxwell equations $(\tilde{g}_{\mu\nu}, F_{\mu\nu})$

- Maxwell equations are automatically fulfilled
- Einstein equations impose conditions on Ω
- Existing theorems do not apply here

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Einstein equations

- Einstein equations in the new spacetime $(\tilde{g}_{\mu\nu}, F_{\mu\nu})$:

$$\tilde{R}_{\mu\nu} = 8\pi \tilde{T}_{\mu\nu}$$

- Transformation properties:

$$\begin{aligned} \tilde{R}_{\mu\nu} &= R_{\mu\nu} - \frac{2}{\Omega}\Omega_{;\mu\nu} + \frac{4}{\Omega^2}\Omega_{,\mu}\Omega_{,\nu} - \frac{1}{\Omega}\Box\Omega g_{\mu\nu} - \frac{1}{\Omega^2}\Omega_{,\rho}\Omega^{,\rho} g_{\mu\nu} \\ \tilde{T}_{\mu\nu} &= \Omega^{-2} T_{\mu\nu} \end{aligned}$$

Conditions on the conformal factor

- Resulting equations:

$$(\Omega^2 - 1)R_{\mu\nu} - 2\Omega\Omega_{;\mu\nu} + 4\Omega_{,\mu}\Omega_{,\nu} - g_{\mu\nu}\Omega_{,\rho}\Omega^{,\rho} = 0 \qquad (1)$$

- Trace $\Rightarrow \Box\Omega = 0$
- Overdetermined system: 10 equations for one function Ω, generally no solution (except for $\Omega^2 \equiv 1$)

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Seed Spacetime

- Most seed spacetimes do not admit non-trivial Ω
- The only suitable family of spacetimes we found so far: *pp*-waves

$$ds^2 = -2H(u, \xi, \bar{\xi})du^2 - 2dudv + 2d\xi d\bar{\xi}$$

Note: Vector field $\mathbf{k} := \partial/\partial v$ is covariantly constant

Conformal Factor

- Eqs. (1) have a non-trivial solution iff $H(u, \xi, \bar{\xi}) = h(u)\xi\bar{\xi}$ in suitable coordinates and $\Omega = \Omega(u)$
- Such *pp*-wave is conformally flat
- Eqs. (1) then reduce to an ODE:

$$\Omega \frac{d^2\Omega}{du^2} - 2\left(\frac{d\Omega}{du}\right)^2 + h(u)(1 - \Omega^2) = 0 \qquad (2)$$

New Spacetime

- $\mathbf{k} := \partial/\partial v$ remains covariantly constant $\Rightarrow$ new spacetime is also a conformally flat *pp*-wave
- By construction: both *pp*-waves have the same $F_{\mu\nu}$
- Question: Is the new spacetime different from the seed?

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(Non)Equivalence

- Explicit choice of seed

$$ds^2 = -4\xi\bar{\xi}du^2 - 2dudv + 2d\xi d\bar{\xi}$$

Note: $R_{\mu\nu} = 4k_\mu k_\nu \Rightarrow R_{\mu\nu;\rho} = 0$

- Explicit solution of eq. (2): $\Omega = \tanh(u)$, i.e.

$$\widetilde{ds}^2 = \tanh^2(u)(-4\xi\bar{\xi}du^2 - 2dudv + 2d\xi d\bar{\xi})$$

Note $\tilde{R}_{\mu\nu} = 4\coth^2(u)k_\mu k_\nu \Rightarrow \tilde{R}_{\mu\nu;\rho} \neq 0$

- Answer: Spacetimes are not equivalent!

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Summary

- We presented a generating method that produces a solution to Einstein-Maxwell equations if a suitable seed spacetime is found
- Once again, *pp*-waves proved to play an important role in context of conformal transformation

- Outlook
  - Are there any other suitable seeds other than *pp*-waves?
  - Can the method be generalized to be less restrictive?