Conformal Yano-Killing Tensors in General Relativity

Jacek Jezierski
Katedra Metod Matematycznych Fizyki
Uniwersytet Warszawski, ul. Hoża 69(74), 00-682 Warszawa

e-mail: Jacek.Jezierski@fuw.edu.pl

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Reports on Mathematical Physics **64** (2009) 205-221

How CYK tensors appear in GR?

- Geometric definition of the asymptotic flat spacetime *strong asymptotic flatness*, which guarantees well defined total angular momentum
- Conserved quantities – asymptotic charges ($\mathscr{I}$, i^0)
- quasilocal mass and "rotational energy" for Kerr black hole

Spacetimes possesing CYK tensor:

- Minkowski (quadratic polynomials)
- (anti)deSitter (natural construction)
- Kerr (type D spacetime)
- Taub-NUT (new symmetric conformal Killing tensors)

Other applications:

- Symmetries of Dirac operator
- Symmetries of Maxwell equations

Conformal Yano–Killing tensors

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field. Contracting the Weyl tensor $W^{\mu\nu\kappa\lambda}$ with $Q_{\mu\nu}$ we obtain a natural object which can be integrated over two-surfaces. The result does not depend on the choice of the surface if $Q_{\mu\nu}$ fulfills the following condition introduced by Penrose

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda}Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \,. \tag{1}$$

one can rewrite equation (1) in a generalized form for n-dimensional spacetime with metric $g_{\mu\nu}$:

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1} g_{\sigma[\lambda}Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \tag{2}$$

or in the equivalent form:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \,. \tag{3}$$

Let us define

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \tag{4}$$

Definition 1. *A skew-symmetric tensor* $Q_{\mu\nu}$ *is a* conformal Yano–Killing tensor *(or simply CYK tensor) for the metric* g *iff* $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.

Other definitions of CYK tensors known also as *Conformal Killing forms* or *Twistor forms*:

A more abstract way with no indices of describing a CYK tensor can be found in literature: Moroianu, Semmelmann or Stepanow, where it is considered as the element of the kernel of the twistor operator

$$Q \to \mathcal{T}_{wist} Q$$

defined as follows:

$$\forall X \;\; \mathcal{T}_{wist} Q(X) := \nabla_X Q - \frac{1}{p+1} X \lrcorner \mathrm{d} Q + \frac{1}{n-p+1} g(X) \wedge \mathrm{d}^* Q \,.$$

Q is a differential p-form on n-dimensional Riemannian manifold.

However, to simplify the exposition, we prefer abstract index notation which also seems to be more popular.

The CYK tensor is a natural generalization of the Yano tensor with respect to the conformal rescalings. More precisely, for any positive scalar function $\Omega > 0$ and for a given metric $g_{\mu\nu}$ we obtain:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = \Omega^{-3}\mathcal{Q}_{\lambda\kappa\sigma}(\Omega^3 Q, \Omega^2 g) \,. \tag{5}$$

The formula (5) and the above definition of CYK tensor gives the following

Theorem 1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$ than $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

It is interesting to notice, that a tensor $A_{\mu\nu}$ — a “square” of the CYK tensor $Q_{\mu\nu}$ defined as follows:

$$A_{\mu\nu} := Q_{\mu}{}^{\lambda} Q_{\lambda\nu}$$

fulfills the following equation:

$$A_{(\mu\nu;\kappa)} = g_{(\mu\nu} A_{\kappa)} \quad \text{with} \quad A_\kappa = \frac{2}{n-1} Q_{\kappa}{}^{\lambda} Q_{\lambda}{}^{\delta}{}_{;\delta} \tag{6}$$

which simply means that the symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor. This can be also described by the following

Theorem 2. *If $Q_{\mu\nu}$ is a skew-symmetric conformal Yano–Killing tensor than $A_{\mu\nu} := Q_{\mu}{}^{\lambda} Q_{\lambda\nu}$ is a symmetric conformal Killing tensor.*

Remark CYK tensor is a solution of the following conformally invariant equation ($n = \dim M = 4$):

$$\left(\Box + \frac{1}{6}\mathcal{R}\right) Q = \frac{1}{2} W(Q, \cdot)$$

$\mathcal{R} := R_{\mu\nu} g^{\mu\nu}$ – scalar curvature, $R_{\mu\nu}$ – symmetric Ricci tensor.

Moreover, if Q is a CYK tensor and the metric is Einstein then

$$K^{\mu} := Q^{\mu\lambda}{}_{;\lambda}$$

is a Killing vector field.

More precisely, one can show

$$K_{(\mu;\nu)} = \frac{n-1}{n-2} R_{\sigma(\mu} Q_{\nu)}{}^{\sigma}$$

which always implies $K^{\mu}{}_{;\mu} = 0$ and the following

Theorem 3. *If $g_{\alpha\beta}$ is an Einstein metric, i.e. $R_{\mu\nu} = \lambda g_{\mu\nu}$, then K^{μ} is a Killing vectorfield.*

Integrability condition

$$Q_{\lambda\kappa}{}^{;\mu}{}_{\mu} + R^{\sigma}{}_{\kappa\lambda\mu}Q^{\mu}{}_{\sigma} + Q_{\sigma\kappa}R^{\sigma}{}_{\lambda} + \frac{2}{n-1}\xi_{(\kappa;\lambda)} + \frac{1}{n-1}g_{\kappa\lambda}\xi^{\mu}{}_{;\mu} = \nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda} - \frac{n-4}{n-1}\xi_{\kappa;\lambda}\,. \quad (7)$$

For $n = \dim M = 4$ eq. (7) implies the following equation for a CYK tensor Q:

$$\nabla^{\mu}\nabla_{\mu}Q_{\lambda\kappa} = R^{\sigma}{}_{\kappa\lambda\mu}Q_{\sigma}{}^{\mu} - R_{\sigma[\kappa}Q_{\lambda]}{}^{\sigma} \quad (8)$$

It is interesting to point out that for compact four-dimensional Riemannian manifolds we have the following

Theorem 4. *Let M be a compact (without boundary) four-dimensional Riemannian manifold; then a two-form Q is a CYK tensor iff*

$$\nabla^{\mu}\nabla_{\mu}Q_{\lambda\kappa} = R^{\sigma}{}_{\kappa\lambda\mu}Q_{\sigma}{}^{\mu} + R_{\sigma[\lambda}Q_{\kappa]}{}^{\sigma}\,.$$

Dowód. We need to show that equation (8) implies $\mathcal{Q}_{\lambda\kappa\mu}(Q,g) = 0$.

We derive

$$\frac{2}{3}\xi_{(\mu;\lambda)} + \frac{1}{3}g_{\mu\lambda}\xi^{\nu}{}_{;\nu} - R_{\sigma(\mu}Q_{\lambda)}{}^{\sigma} + \frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\lambda\sigma\mu} = 0\,, \quad 4\xi^{\mu}{}_{;\mu} + \nabla^{\sigma}\mathcal{Q}_{\nu\sigma\mu}g^{\mu\nu} = 0\,,$$

which together with

$$2\xi^{\mu}{}_{;\mu} = Q^{\lambda\kappa}{}_{;\lambda\kappa} - Q^{\lambda\kappa}{}_{;\kappa\lambda} = 2Q^{\kappa\sigma}R_{\sigma\kappa} = 0 \tag{9}$$

and (7) gives

$$\nabla^{\mu}\nabla_{\mu}Q_{\lambda\kappa} + R^{\sigma}{}_{\kappa\lambda\mu}Q^{\mu}{}_{\sigma} + R_{\sigma[\kappa}Q_{\lambda]}{}^{\sigma} = \nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda} + \frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\kappa\sigma\lambda}\,. \tag{10}$$

Contracting the above equality with Q and assuming equation (8) we get

$$\begin{aligned} 0 = \left(\nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda} + \frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\lambda\sigma\kappa}\right)Q^{\kappa\lambda} &= \nabla^{\mu}\left(\mathcal{Q}_{\mu\kappa\lambda}Q^{\kappa\lambda}\right) - \mathcal{Q}_{\mu\kappa\lambda}\nabla^{\mu}Q^{\kappa\lambda} \\ &= \nabla^{\mu}\left(\mathcal{Q}_{\mu\kappa\lambda}Q^{\kappa\lambda}\right) + \frac{1}{2}\mathcal{Q}_{\lambda\kappa\mu}\mathcal{Q}^{\lambda\kappa\mu}\,. \end{aligned} \tag{11}$$

Finally, we integrate the above formula over M, a total divergence drops out, and the integral $\int_M \sqrt{\det g}\,\mathcal{Q}_{\lambda\kappa\mu}\mathcal{Q}^{\lambda\kappa\mu}$ vanishes. This implies $\mathcal{Q}^{\lambda\kappa\mu} = 0$. □

A similar result holds for a p-form Q in $2p$-dimensional M.

Let us restrict ourselves to four-dimensional manifold ($n = 4$). The Hodge-dual of $Q_{\mu\nu}$ defined as follows

$$*Q_{\kappa\lambda} = \frac{1}{2}\varepsilon_{\kappa\lambda}{}^{\mu\nu}Q_{\mu\nu} \ .$$

gives also a two-form. Multiplying CYK equation

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}K_\kappa - g_{\kappa(\lambda}K_{\sigma)}\right)$$

by $\frac{1}{2}\varepsilon^{\alpha\beta\lambda\kappa}$ we get:

$$*Q_{\alpha\beta;\sigma} = \frac{2}{3}g_{\sigma[\alpha}\chi_{\beta]} + \frac{1}{3}\varepsilon_{\alpha\beta\sigma\kappa}K^\kappa \ , \tag{12}$$

where $\chi_\mu := *Q^\nu{}_{\mu;\nu}$ and $K_\mu = Q^\nu{}_{\mu;\nu}$. Multiplying the above equality by $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}$, we obtain a similar formula:

$$Q_{\mu\nu;\sigma} = \frac{2}{3}g_{\sigma[\mu}K_{\nu]} - \frac{1}{3}\varepsilon_{\mu\nu\sigma\beta}\chi^\beta \ . \tag{13}$$

Finally, symmetrization of indices α and σ in (12) gives:

$$*Q_{\alpha\beta;\sigma} + *Q_{\sigma\beta;\alpha} = \frac{2}{3}\left(g_{\sigma\alpha}\chi_\beta - g_{\beta(\alpha}\chi_{\sigma)}\right) ,$$

which implies the following

Theorem 5. *$Q_{\mu\nu}$ is a CYK tensor iff $*Q_{\mu\nu}$ is a CYK tensor.*

In particular, it implies that Einstein manifolds possessing non-trivial CYK tensors should admit two Killing fields $K_\mu = Q^\nu{}_{\mu;\nu}$ and $\chi_\mu = *Q^\nu{}_{\mu;\nu}$. They sometimes vanish which simply means that CYK tensor or its dual is a usual Yano tensor.

For any two-form $Q_{\mu\nu}$ we have the following identity:

$$\nabla_\lambda \left(W^{\mu\lambda\alpha\beta} Q_{\alpha\beta}\right) = \frac{2}{3} W^{\mu\lambda\alpha\beta} \mathcal{Q}_{\alpha\beta\lambda} \,. \tag{14}$$

More precisely,

$$\nabla_\lambda \left(W^{\mu\lambda\alpha\beta} Q_{\alpha\beta}\right) = \nabla_\lambda \left(W^{\mu\lambda\alpha\beta}\right) Q_{\alpha\beta} + W^{\mu\lambda\alpha\beta} \nabla_\lambda \left(Q_{\alpha\beta}\right) .$$

and first term vanishes for spin-2 field W, but the second one equals to right-hand side of (14) because of the symmetries of W. This implies that for any CYK tensor $Q_{\mu\nu}$ we have

$$\int_{\partial V} W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa} \mathrm{d}S_{\mu\nu} = \int_V (W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa})_{;\nu} \mathrm{d}\Sigma_\mu =$$

$$= \int_V (W^{\mu\nu\lambda\kappa}{}_{;\nu} Q_{\lambda\kappa} + W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa;\nu}) \mathrm{d}\Sigma_\mu = 0 \,.$$

The above equality implies that the flux of the quantity $W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa}$ through any two closed two-surfaces S_1 and S_2 is the same if there is a three-volume V between them (i.e. if $\partial V = S_1 \cup S_2$). We define the charge corresponding to the specific CYK tensor Q as the value of this flux

Spin-2 field

Let us start with the standard formulation of a spin-2 field $W_{\mu\alpha\nu\beta}$ in Minkowski spacetime equipped with a flat metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. The field W can be also interpreted as a Weyl tensor for linearized gravity.

Definition 2. *The following properties:*

$$W_{\mu\alpha\nu\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]}, \; W_{\mu[\alpha\nu\beta]} = 0, \; \eta^{\mu\nu} W_{\mu\alpha\nu\beta} = 0 \tag{15}$$

can be used as a definition of spin-2 field W.

The $*$–operation defined as

$$({}^*W)_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}\,, \quad (W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$$

has the following properties:

$$({}^*W^*)_{\alpha\beta\gamma\delta} = \frac{1}{4}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\gamma\delta}\,, {}^*W = W^*, \; {}^*({}^*W) = {}^*W^* = -W\,,$$

where $\varepsilon_{\mu\nu\gamma\delta}$ is a Levi–Civita skew-symmetric tensor and ${}^*W$ is called dual spin-2 field.

The above formulae are also valid for general Lorentzian metrics.

Moreover, Bianchi identities play a role of field equations and we have the following

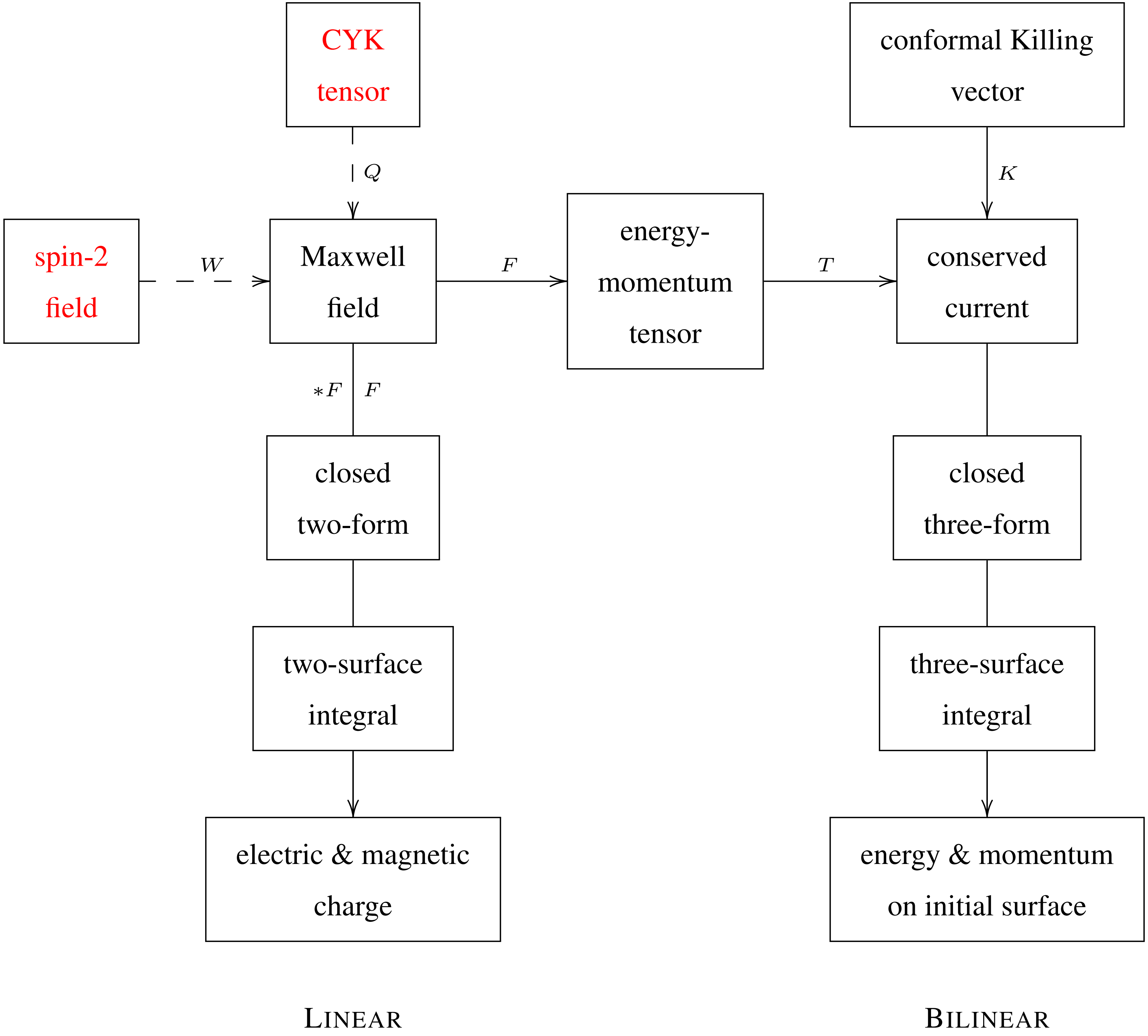
Lemma 1. *Field equations*

$$\nabla_{[\lambda} W_{\mu\nu]\alpha\beta} = 0 \tag{16}$$

are equivalent to

$$\nabla^{\mu} W_{\mu\nu\alpha\beta} = 0 \;\; or \;\; \nabla_{[\lambda} {}^{*}W_{\mu\nu]\alpha\beta} = 0 \;\; or \;\; \nabla^{\mu}\, {}^{*}W_{\mu\nu\alpha\beta} = 0 \; .$$

The equations in the above Lemma are also valid for any Ricci flat metric and its Weyl tensor.


CYK tensor
conformal Killing vector
Q
K
spin-2 field
W
Maxwell field
F
energy-momentum tensor
T
conserved current
*F
F
closed two-form
closed three-form
two-surface integral
three-surface integral
electric & magnetic charge
energy & momentum on initial surface
LINEAR
BILINEAR