Algebraic Rainich conditions for the fourth rank tensor V

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Abstract

Algebraic conditions on the Ricci tensor in the Rainich-Misner-Wheeler unified field theory are known as the Rainich conditions. Penrose and more recently Bergqvist and Lankinen made an analogy from the Ricci tensor to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$, a certain fourth rank tensor quadratic in the Weyl curvature, which also satisfies algebraic Rainich-like conditions. However, we found that not only does the tensor $B_{\alpha\beta\mu\nu}$ fulfill these conditions, but so also does our recently proposed tensor $V_{\alpha\beta\mu\nu}$, which has many of the desirable properties of $B_{\alpha\beta\mu\nu}$. For the quasilocal small sphere limit restriction, we found that there are only two fourth rank tensors, $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$, which form a basis for good energy expressions. Both of them have the completely trace free and causal properties, these two form necessary and sufficient conditions. Surprisingly either completely traceless or causal is enough to fulfill the algebraic Rainich conditions.

1 Introduction

We define the 4th rank Bel-Robinson tensor, proposed in 1958, as follows

$$B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + *R_{\alpha\lambda\mu\sigma} * R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma}$$
$$= R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\mu\nu}\mathbf{R}^2. \quad (1)$$

where $*R_{\alpha\lambda\sigma\tau}$ is the dual of $R_{\alpha\lambda\sigma\tau}$ and $\mathbf{R}^2 = R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}$. Here we propose an unique alternative 4th rank tensor for the non-negative gravitational energy in the quasilocal limit

$$V_{\alpha\beta\mu\nu} = B_{\alpha\beta\mu\nu} + W_{\alpha\beta\mu\nu} \neq B_{\alpha\beta\mu\nu}, \quad (2)$$

in general, where

$$W_{\alpha\beta\mu\nu} := \frac{3}{2}(R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma}) + \frac{1}{8}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - 2g_{\alpha\beta}g_{\mu\nu})\mathbf{R}^2, \quad (3)$$

Analog of the electric E_{ab} and magnetic part H_{ab} are defined in terms of the Weyl tensor

$$E_{ab} := C_{a0b0}, \quad H_{ab} := *C_{a0b0}, \quad a, b = 1, 2, 3. \quad (4)$$

The positive gravitation energy density for both $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are

$$B_{0000} \equiv V_{0000} = E_{ab}E^{ab} + H_{ab}H^{ab} \geq 0. \quad (5)$$

In the small region $P_\mu \propto B_{\mu000} = V_{\mu000} = (E_{ab}E^{ab} + H_{ab}H^{ab},\ 2\epsilon_{cab}E^{ad}H^{b}{}_{d})$.
Since we want $P^0 \geq |\vec{P}| \geq 0$, then causal condition $E_{ab}E^{ab} + H_{ab}H^{ab} \geq 2\epsilon_{cab}E^{ad}H^{b}{}_{d}$

2 Algebraic Rainich conditions

In 1925 Rainich proposed a unified field theory for the geometrization of the electromagnetic field. The basic idea is from the Einstein-Maxwell field equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi\left(F_{\alpha\lambda}F_{\beta}{}^{\lambda} - \frac{1}{4}g_{\alpha\beta}F_{\rho\lambda}F^{\rho\lambda}\right). \tag{6}$$

where $R_{\alpha\beta}$ is the 2nd rank Ricci tensor and $F_{\alpha\beta}$ is the electromagnetic field tensor. Taking the trace of (6), one finds $R^{\alpha}{}_{\alpha} = R = 0$. Then rewrite (6)

$$R_{\alpha\beta} = -8\pi\left(F_{\alpha\sigma}F_{\beta}{}^{\sigma} - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{7}$$

Using this, one can obtain the following

$$R_{\alpha\lambda}R_{\beta}{}^{\lambda} = \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda}R^{\rho\lambda}. \tag{8}$$

Therefore the algebraic Rainich conditions can be written as

$$R_{\alpha\lambda}R_{\beta}{}^{\lambda} = \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda}R^{\rho\lambda}, \qquad R^{\alpha}{}_{\alpha} = 0, \qquad R_{00} \geq 0, \tag{9}$$

where the last condition is to ensure positive energy.

3 Fourth rank Algebraic Rainich conditions

Penrose replaced the Ricci tensor by stress tensor $T_{\alpha\beta}$ using the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\, T_{\mu\nu} \tag{10}$$

Rewrite the algebraic Rainich conditions as follows

$$T_{\alpha\lambda}T_{\beta}{}^{\lambda} = \frac{1}{4}g_{\alpha\beta}T_{\rho\lambda}T^{\rho\lambda}, \qquad T^{\lambda}{}_{\lambda} = 0, \qquad T_{00} \geq 0, \tag{11}$$

Furthermore, analog from 2nd to 4th rank tensor, $B_{\alpha\beta\mu\nu}$ is the first tensor satisfying the algebraic Rainich conditions [Penrose; Bergqvist and Lankinen CQG 2004 **21** 3499].

$$B_{\alpha\lambda\sigma\tau}B_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau}, \quad 0 = B^{\rho}{}_{\rho\sigma\tau} = B^{\rho}{}_{\sigma\rho\tau} = \ldots, \quad B_{0000} \geq 0, \tag{12}$$

where $B_{\alpha\lambda\sigma\tau} \equiv B_{(\alpha\lambda\sigma\tau)}$ (i.e., completely symmetric). Moreover, our recently proposed $V_{\alpha\beta\mu\nu}$ also satisfies these algebraic Rainich conditions

$$V_{\alpha\lambda\sigma\tau}V_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}, \quad 0 = V^{\rho}{}_{\rho\sigma\tau} = V^{\rho}{}_{\sigma\rho\tau} = \ldots, \quad V_{0000} \geq 0, \tag{13}$$

where $V_{(\alpha\lambda)(\sigma\tau)} \equiv V_{(\sigma\tau)(\alpha\lambda)}$ and $V^{\rho}{}_{\rho\sigma\tau} \equiv 0 \equiv V^{\rho}{}_{\sigma\rho\tau}$, but $V_{\alpha\lambda\sigma\tau} \neq V_{(\alpha\lambda\sigma\tau)}$.

Footnote: We found that the first condition is an identity for (12) and (13). The detail in on next slide.

4 Two basic components of the fourth rank tensor

Edgar and Wingbrant [JMP 2003] found the one-quarter identity

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\pi}{}^{\lambda}{}_{\gamma}R^{\sigma\pi\tau\gamma} = \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\pi}{}^{\lambda}{}_{\gamma}R^{\sigma\pi\tau\gamma}, \tag{14}$$

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\pi}{}^{\tau}{}_{\gamma}R^{\sigma\pi\lambda\gamma} = \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\pi}{}^{\tau}{}_{\gamma}R^{\sigma\pi\lambda\gamma}. \tag{15}$$

We found that the above two are not the only cases that satisfy the one-quarter property, but for any combination

$$M_{\alpha\lambda\sigma\tau}M_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta\mu\nu}M_{\rho\lambda\sigma\tau}M^{\rho\lambda\sigma\tau}, \tag{16}$$

where $M_{\alpha\lambda\sigma\tau}$ is any tensor quadratic in the Riemann curvature, non-vanishing in vacuum.

For any fourth rank tensor, Deser [arXiv:gr-qc/9901007] stated that there are only two basic components. We use the following as the representation:

$$(R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau})^2, \quad (R_{\rho\lambda\sigma\tau} * R^{\rho\lambda\sigma\tau})^2. \tag{17}$$

In particular [M Iihoshi and S V Ketov, Advances in High Energy Physics Volume 2008],

$$\underline{4B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} = (R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau})^2 + (R_{\rho\lambda\sigma\tau} * R^{\rho\lambda\sigma\tau})^2} \tag{18}$$

Moreover, we found

$$2B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} = (R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau})^2 - 2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\tau\mu\nu}. \tag{19}$$

The Deser paper does not contain ‘2’ at the last term.

5 Two basic components of the fourth rank tensor

Edgar and Wingbrant [JMP 2003] found the one-quarter identity

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\pi}{}^{\lambda}{}_{\gamma}R^{\sigma\pi\tau\gamma} = \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\pi}{}^{\lambda}{}_{\gamma}R^{\sigma\pi\tau\gamma}, \tag{20}$$

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\pi}{}^{\tau}{}_{\gamma}R^{\sigma\pi\lambda\gamma} = \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\pi}{}^{\tau}{}_{\gamma}R^{\sigma\pi\lambda\gamma}. \tag{21}$$

We found that the above two are not the only cases that satisfy the one-quarter property, but for any combination

$$M_{\alpha\lambda\sigma\tau}M_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta\mu\nu}M_{\rho\lambda\sigma\tau}M^{\rho\lambda\sigma\tau}, \tag{22}$$

where $M_{\alpha\lambda\sigma\tau}$ is any tensor quadratic in the Riemann curvature, non-vanishing in vacuum.

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$$\underline{4B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} = (R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau})^2 + (R_{\rho\lambda\sigma\tau} * R^{\rho\lambda\sigma\tau})^2} = 16R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\pi}{}^{\sigma}{}_{\gamma}R^{\lambda\pi\tau\gamma}. \tag{24}$$

Moreover, we found

$$2B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} = (R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau})^2 - 2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\tau\mu\nu}. \tag{25}$$

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6 Completely traceless imply positivity or causal

There are 4 basic quadratic Riemann tensors in the small region limit [Deser CQG 1999], we use our own notation

$$\tilde{B}_{\alpha\beta\mu\nu} = R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma}, \quad \tilde{K}_{\alpha\beta\mu\nu} = R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma},$$
$$\tilde{S}_{\alpha\beta\mu\nu} = R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma}, \quad \tilde{T}_{\alpha\beta\mu\nu} = -\frac{1}{8}g_{\alpha\beta}g_{\mu\nu}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}. \tag{26}$$

Consider the combination for totally traceless:

$$0 = a_1\tilde{B}^{\alpha}{}_{\alpha\mu\nu} + a_2\tilde{S}^{\alpha}{}_{\alpha\mu\nu} + a_3\tilde{K}^{\alpha}{}_{\alpha\mu\nu} + a_4\tilde{T}^{\alpha}{}_{\alpha\mu\nu} = \frac{1}{2}(a_1 + a_2 - a_4)g_{\mu\nu}\mathbf{R}^2, \tag{27}$$
$$0 = a_1\tilde{B}^{\alpha}{}_{\mu\alpha\nu} + a_2\tilde{S}^{\alpha}{}_{\mu\alpha\nu} + a_3\tilde{K}^{\alpha}{}_{\mu\alpha\nu} + a_4\tilde{T}^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{8}(a_1 - 2a_2 + 3a_3 - a_4)g_{\mu\nu}\mathbf{R}^2, \tag{28}$$

where a_1 to a_4 are constants. The solutions are

$$a_4 = a_1 + a_2, \quad a_2 = a_3. \tag{29}$$

Consider the linear combination of these 4 fundamental tensors, we found $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are the unique tensors that have the completely traceless property and imply positivity

$$\begin{aligned} & a_1\tilde{B}_{\alpha\beta\mu\nu} + a_2\tilde{S}_{\alpha\beta\mu\nu} + a_3\tilde{K}_{\alpha\beta\mu\nu} + a_4\tilde{T}_{\alpha\beta\mu\nu} \\ = \; & a_1(\tilde{B}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) + a_2(\tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) \\ = \; & a_1B_{\alpha\beta\mu\nu} + a_2V_{\alpha\beta\mu\nu}. \end{aligned} \tag{30}$$

7 Positive or causal implies completely traceless

Counter example for satisfying positive energy but not fulfilling completely traceless,

$$X_{\alpha\beta\mu\nu} = R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma}, \quad X_{0000} = E_{ab}E^{ab} \geq 0, \quad X^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{4}g_{\mu\nu}\mathbf{R}^2 \neq 0. \tag{31}$$

Returning back to causal, consider the energy-momentum integral in a quasilocal small sphere with constant time evolution of the hypersurface. Note that the fourth rank tensor $X_{\alpha\beta\mu\nu}$ needs to be symmetric at the last two indices because of the small sphere limit

$$\begin{aligned} N^{\mu}P_{\mu} &= \int N^{\mu}X^{\rho}{}_{\mu\xi\kappa}x^{\xi}x^{\kappa}\eta_{\rho} = \int N^{\mu}X^{0}{}_{\mu\xi\kappa}x^{\xi}x^{\kappa}\eta_{0} = \int N^{\mu}X^{0}{}_{\mu ij}x^{i}x^{j}dV \\ &= \int N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{r^2}{3}dV = \int N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{r^2}{3}4\pi r^2 dr = N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{4\pi r^5}{15} \\ &= N^{\mu}\left(X^{0}{}_{\mu\alpha}{}^{\alpha} - X^{0}{}_{\mu 0}{}^{0}\right)\frac{4\pi r^5}{15} = N^{\mu}X^{0}{}_{\mu 00}\frac{4\pi r^5}{15}, \end{aligned} \tag{32}$$

where we made the assumption that $X_{0\mu\alpha}{}^{\alpha}$ vanishes and fulfills causal. Consider the requirement for the energy-momentum being future pointing and non-spacelike (i.e., causal) in the small sphere limit :

$$\begin{aligned} & a_1\tilde{B}_{\mu 0l}{}^{l} + a_2\tilde{S}_{\mu 0l}{}^{l} + a_3\tilde{K}_{\mu 0l}{}^{l} + a_4\tilde{T}_{\mu 0l}{}^{l} \\ =\ & a_1(-2E_{ab}E^{ab} + 4H_{ab}H^{ab},\ 2\epsilon_{cab}E^{ad}H^{b}{}_{d}) + a_2(-4E_{ab}E^{ab} + 4H_{ab}H^{ab},\ 0) \\ & +a_3(2E_{ab}E^{ab},\ 2\epsilon_{cab}E^{ad}H^{b}{}_{d}) + a_4(3E_{ab}E^{ab} - 3H_{ab}H^{ab},\ 0) \\ =\ & (-2a_1 - 4a_2 + 2a_3 + 3a_4)E_{ab}E^{ab} + (4a_1 + 4a_2 - 3a_4)H_{ab}H^{ab} + (2a_1 + 2a_3)\epsilon_{cab}E^{ad}H^{b}{}_{d}. \end{aligned}$$

Causal requires the magnitude of $E_{ab}E^{ab}$ and $H_{ab}H^{ab}$ to be the same and the energy is greater than or equal to the momentum. After some simple algebra, we found

$$a_1 \tilde{B}_{\mu 0 l}{}^{l} + a_2 \tilde{S}_{\mu 0 l}{}^{l} + a_3 \tilde{K}_{\mu 0 l}{}^{l} + a_4 \tilde{T}_{\mu 0 l}{}^{l} = a_1 B_{\mu 000} + a_3 V_{\mu 000} = (a_1 + a_3) B_{\mu 000}, \tag{33}$$

where we require $a_1 + a_3 \geq 0$.

Hence, the completely traceless and causal properties form necessary and sufficient conditions. This means we can further simplify the algebraic Rainich conditions for a fourth rank tensor; as far as the quasilocal small sphere limit is concerned, we only need the completely trace free condition or positivity (i.e., causal). This is an interesting result which is valid in the quasilocal small sphere region.

8 Conclusion

Either completely traceless or causal is enough to fulfill the algebraic Rainich conditions.

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