U(N) Tools for Loop Quantum Gravity

Etera Livine

Ecole Normale Supérieure de Lyon - CNRS

Spanish Relativity Meeting - ERE 2010

September 2010 in Granada

Started with F. Girelli ,

then work with L. Freidel and with E. Borja , J. Diaz-Polo & I. Garay

arXiv:gr-qc/0501075, 0911.3553, 1005.2090, 1006.2451, 1006.5666, and more coming soon!

Aim: Use new U(N) tools to probe the structure of the intertwiner space of Loop Quantum Gravity and reformulate the LQG dynamics.

A U(N) Action on the Space of SU(2) Intertwiners

1. Characterizes Intertwiner Spaces as U(N) Irreps
   $\hookrightarrow$ Intertwiner counting and Black Hole entropy in LQG
2. The Intertwiner Space as a L^2 space with
   $\hookrightarrow$ Creation/annihilation operators
   $\hookrightarrow$ Semi-classical coherent intertwiner states
3. Improves the geometrical interpretation of Intertwiners
   $\hookrightarrow$ Reconstructing spin networks from gluing classical poyhedra
4. Provides new operators for the LQG Dynamics
   $\hookrightarrow$ Reformulation of Holonomy/Flux operators

The Loop Gravity Quantization Scheme

Loop Quantum Gravity = Canonical Quantization of GR

Based on a 3+1 formalism $\mathcal{M} = \mathbb{R} \times \Sigma_{3d}$

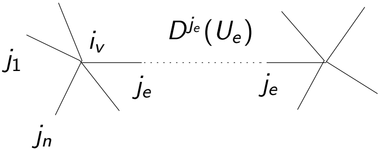
- GR as a Gauge Theory : $\{A^a_i(x), E^j_b(y)\} = \gamma \delta^j_i \delta^a_b \delta^{(3)}(x-y)$
  - $\hookrightarrow$ Triad field E^i_a gives 3-metric $h^{ij} = E^i_a E^j_a$
  - $\hookrightarrow$ Ashtekar-Barbero $\mathfrak{su}(2)$ connection $A^a_i = \Gamma^a_i(E) + \gamma K^a_i$
  - $\hookrightarrow$ Immirzi parameter γ
- GR as a Constrained System : $H = \Lambda^a \mathcal{G}_a + N_i \mathcal{H}^i + N\mathcal{H}$
  - $\hookrightarrow$ SU(2) gauge transformations $\mathcal{G}_a \sim d_A E$
  - $\hookrightarrow$ 3d space diffeomorphisms (on-shell) generated by $\mathcal{H}^i \sim EF[A]$
  - $\hookrightarrow$ Hamiltonian constraint encoding time evolution $\mathcal{H} \sim EEF[A]$

Spin Network States and Intertwiners

A straightforward quantization procedure:

- Wave-functions $\psi(A)$ with triad acting as $\hat{E} \sim i\gamma\frac{\partial}{\partial A}$
- Choose $\mathrm{SU}(2)$-inv cylindrical functions $\psi(\{U_e\}_{e\in\Gamma})$ depending on holonomies along the edges of a graph Γ
- Identify the Spin Network basis of LQG quantum states: Label edges with **spins** j_e, put **intertwiners** i_v at vertices and glue them with the holonomies along the edges.

$$\psi^{(\Gamma)}_{j_e,i_v}(U_e) \equiv \mathrm{Tr} \otimes_v i_v \otimes_e D^{j_e}(U_e)$$



- Then implement $\mathcal{H}^i, cH$ constraints on these states.

Intertwiners ?

At each vertex v, the intertwiner intertwines the representations living on the in-coming edges:

Intertwiner i_v $= \mathrm{SU}(2)$-inv tensor in $j_1^{(v)} \otimes .. \otimes j_n^{(v)}$ = singlet state

Geometrical Interpretation

We build geometrical operators out of the triad $\hat{E}$. It provides spin networks with an interpretation as quantized discrete geometries .

- Elementary surface $\mathcal{A}$ dual to edge e
  $\Rightarrow \mathcal{A}^2 = \gamma^2 \vec{J}_e \cdot \vec{J}_e = \gamma^2 j_e(j_e + 1)$ easy ✓
  Only depends on the spins.
- Chunk of volume $\mathcal{V}$ dual to vertex v, picks triplets of edges
  $\Rightarrow \mathcal{V}^2 = \gamma^3 \vec{J}_{e_1} \cdot (\vec{J}_{e_2} \wedge \vec{J}_{e_3})$ harder. . .
  Depends on the intertwiners.

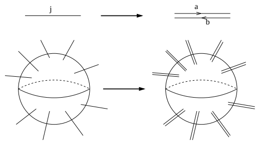
Intertwiners = basic building block for quantum geometry in LQG

The $\mathrm{U}(N)$ structure of the intertwiner space

The Object: the space of intertwiners with N legs i.e of $\mathrm{SU}(2)$-inv states in $V^{j_1} \otimes .. \otimes V^{j_N}$ for arbitrary values of spins $j_i \in \mathbb{N}/2$.
The first tool: The Schwinger representation of $\mathfrak{su}(2)$ in term of harmonic oscillators

We introduce a_i, b_i for each leg i.

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij},$$



$$J_i^z = \frac{1}{2}(a_i^\dagger a_i - b_i^\dagger b_i) \to m_i$$

$$J_i^+ = a_i^\dagger b_i, \; J_i^- = a_i b_i^\dagger,$$

$$E_i = (a_i^\dagger a_i + b_i^\dagger b_i) \to j_i$$

The U(N) structure of the intertwiner space

We build SU(2)-invariant quadratic operators acting on pairs of legs:

$$E_{ij} \equiv (a_i^\dagger a_j + b_i^\dagger b_j)$$

They shift the spins j_i and form a $\mathfrak{u}(N)$-algebra :

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

The diagonal components give the spins living on the edges:

$$\begin{aligned} E_{ii} &= E_i = \text{ spin } 2j_i, \\ E &= \sum E_i = 2 \times \sum_i j_i = 2 \times \text{ total area}, \\ [E, E_{ij}] &= 0. \end{aligned}$$

The U(N)-action on Intertwiners

The basic conclusion: unitary group U(N) acts on intertwiners.

After a little more work, U(N) acts on the space of intertwiners with fixed N (obviously) and with fixed total boundary area (U(1) Casimir E):

$$R_N^J = \bigoplus_{\sum_i^N j_i = J} \mathrm{Inv}[V^{j_1} \otimes .. \otimes V^{j_N}]$$

Irreducible representations with highest weight vector satisfying:

$$E_1|v\rangle = E_2|v\rangle = J|v\rangle, \quad E_{k\geq 3}|v\rangle = 0 \Rightarrow \text{bivalent intertwiner.}$$

The operators E_{ij} shift the spins on the legs and allow to go from this highest weight vector to arbitrary intertwiner states.

More SU(2)-invariant operators !

We identify a new set of SU(2)-invariant quadratic operators:

$$F_{ij} \equiv a_i b_j - a_j b_i, \quad F_{ij} = -F_{ji}.$$

They form a closed algebra with the E_{ij}. They are invariant under SU(2) but do not preserve the total area!!

$$[E, F_{ij}] = -2F_{ij}, \qquad [E, F^\dagger_{ij}] = +2F^\dagger_{ij}$$

⇒ annihilation/creation operators

They allow to shift between U(N) irreps and provide the space of intertwiners with a Fock space interpretation:

$$R_N = \bigoplus_{\{j_i\}} \mathrm{Inv}[V^{j_1} \otimes .. \otimes V^{j_N}] = \bigoplus_{J \in \mathbb{N}} R_N^J.$$

What has been done with the U(N) framework?

- The original objective: We replaced the scalar product operators $\vec{J}_i \cdot \vec{J}_j$ by another set of $\mathrm{SU}(2)$-invariant operators E_{ij}, which form a closed $\mathfrak{u}(N)$ algebra.
  $\hookrightarrow$ Useful to define semi-classical states.
- $\mathrm{U}(N)$ = unitary deformations of intertwiners at fixed area.
  $\hookrightarrow$ Useful to define evolution for LQG?
- Intertwiner Counting from the dimensions of $\mathrm{U}(N)$ irreps
  $\hookrightarrow$ Applied to LQG black hole entropy calculations
- Constructing coherent intertwiners using the creation operators $(F^\dagger)^J$
  $\hookrightarrow$ Coherent states under $\mathrm{U}(N)$ transformations
  $\hookrightarrow$ Peak the values of scalar products
  $\hookrightarrow$ Useful to define semi-classical spin network states

Going further with U(N) tools: Two developments

1. The intertwiner space as a L^2 space :

$$R_N = L^2_{holo}\left(\frac{\mathrm{U}(N)}{\mathrm{U}(N-2)\times\mathrm{SU}(2)}\right) = L^2_{holo}\left(\frac{\mathbb{C}^{2N}}{\mathbb{C}\times\mathrm{GL}(2,\mathbb{C})}\right)$$

$\rightsquigarrow$ as wave-functions on U(N) or of N spinors.
$\Rightarrow$ Allows to truly view intertwiners as quantized polyhedra.

2. Application to the 2-vertex model :
Truncation of QG to spin networks living on the 2-vertex graph (proposed by Rovelli & Vidotto)
$\hookrightarrow$ Established a dictionary between U(N)-operators and holonomy /grasping operators of LQG
$\hookrightarrow$ Use the U(N) symmetry to average over all intertwiners with same total boundary area, and effectively reduce to isotropic states i.e impose spherical symmetry .
$\Rightarrow$ Reducing from LQG to (L)QC at the quantum level...

Where are we going with the U(N) framework for Loop Quantum Gravity?

Some direction for the future (of the U(N) formalism):

- A reformulation of LQG dynamics in terms of spinors and U(N) operators acting directly on the vertices/intertwiners of the spin network states?
- A better understanding of how to write semi-classical states for the quantum geometry?
- Progress on understanding how to deform spin network states using U(N) transformations $\rightsquigarrow$ discrete diffeomorphisms?
- Implement the symmetry reduction to quantum Cosmology (and possibly other symmetry reduced settings in general relativity) directly at the quantum level from Loop Quantum Gravity $\rightarrow$ an explicit bridge between LQG and LQC?
- And now to Iñaki 's talk!