Stochastic Inflation in Compact Extra Dimensions

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work with J. Martin and J. Yokoyama,
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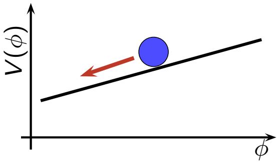
Slow Roll Trajectory vs. Quantum Kicks

classical drift:

$$3H\dot{\phi} + V_\phi \approx 0$$
$$\frac{\kappa}{3} V(\phi) \approx H^2$$



typical time interval
$\Delta t = 1/H$:

$$\boldsymbol{\Delta\phi_{cl} = -\frac{V'}{3H}\Delta t = -\frac{V'}{3H^2}}$$

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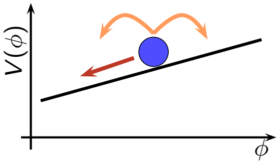


quantum kicks:

$$\boldsymbol{\Delta\phi_{qu} = \frac{H}{2\pi}}$$

If $\Delta\phi_{qu} \simeq \Delta\phi_{cl}$, classical trajectory receives quantum corrections.

"stochastic inflation"

Dual Descriptions

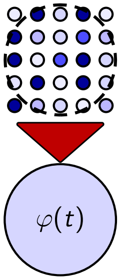
separate ϕ into long / short wavelength modes

$$\phi(\vec{x}, t) = \varphi(t) \qquad \text{long} = \text{smoothed}$$

$$+ \underbrace{\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{3/2}} \Theta(k - \sigma a H) \left[\hat{a}_k \, \delta\phi_k(t) \, e^{-i\vec{k}\cdot\vec{x}} + \mathrm{h.c.} \right]}_{\text{short} = \text{noise}}$$

Starobinsky & Yokoyama (1994), Martin & Musso (2006), Kühnel & Schwarz (2008,2009,2010) *etc.*

scale $\propto 1/\sigma H$



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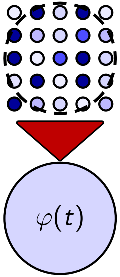

Smoothed field $\underbrace{\varphi(t)}_{\text{long}}$ is a stochastic process w/ gaussian white noise $\underbrace{\xi(t)}_{\text{short}}$.

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$$\dot{\varphi} = -\frac{V'}{3H} + \frac{H^{3/2}}{2\pi}\,\xi(t)$$

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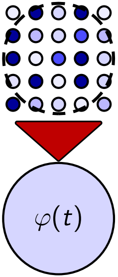

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Fokker Planck:

$$\frac{\partial P(\varphi,t)}{\partial t} = \frac{\partial}{\partial \varphi} \left\{ \frac{H^{3/2}}{8\pi^2} \frac{\partial}{\partial \varphi} \left[H^{3/2} P(\varphi, t) \right] + \frac{V'}{3H} P(\varphi, t) \right\}$$

Perturbative Solution (Martin & Musso 2006)

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probability to find field value φ at time t in single domain

- use $\left\langle \delta\varphi_1^2 \right\rangle$ and $\langle \delta\varphi_2 \rangle$ to determine $P(\varphi, t)$ in the gaussian approximation

$$\left\langle \delta\varphi_1^2 \right\rangle = \frac{\kappa}{2}\left(\frac{H'}{2\pi}\right)^2 \int_{\varphi_{\rm cl}}^{\phi_{\rm in}} \mathrm{d}\phi \left(\frac{H}{H'}\right)^3$$

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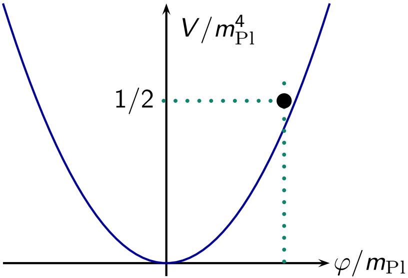
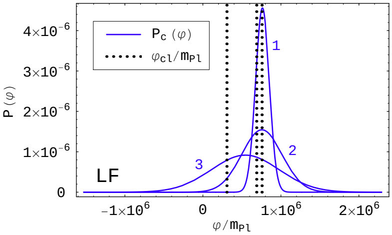
$$\mathbf{P}(\varphi, \mathbf{t}) = \frac{1}{\sqrt{2\pi\left\langle \delta\varphi_1^2 \right\rangle}} \exp\left[-\frac{(\varphi - \langle\varphi\rangle)^2}{2\left\langle \delta\varphi_1^2 \right\rangle}\right]$$

where $\langle\varphi\rangle \equiv \varphi_{\rm cl} + \langle\delta\varphi_2\rangle$

Example

chaotic inflation with $V(\varphi) = \frac{m^2}{2}\,\varphi^2$, large $\phi_{\rm in}/m_{\rm Pl}$

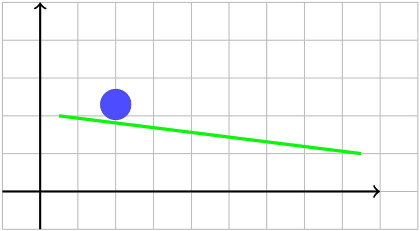


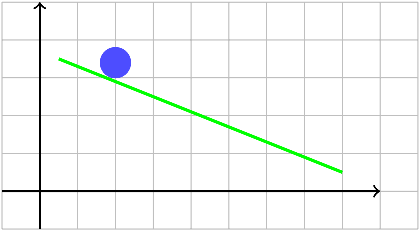


Martin & Musso (2006)

k-Inflation

two ways to drive inflation:





flat potential $\mathcal{L} = \frac{\dot{\varphi}^2}{2} - V(\varphi)$

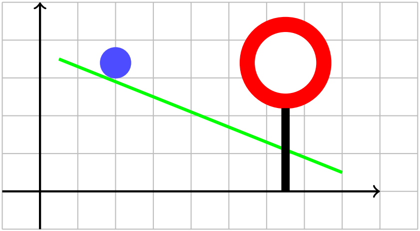
modified kinetic term $\mathcal{L} = P(X, \varphi)$
where $X = -\frac{g^{\mu\nu}}{2}\partial_\mu\varphi\partial_\nu\varphi$

Armendariz-Picon, Damour and Mukhanov (1999)
Garriga and Mukhanov (1999)

k-Inflation from String Theory: Dirac Born Infeld (DBI)

effective 4d DBI inflaton action:

$$S = -\int \mathrm{d}^4x\sqrt{-g}\Big[V(\varphi) - T(\varphi) + T(\varphi)\sqrt{1+\frac{1}{T(\varphi)}\,g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}\Big]$$



$V(\varphi)$: potential
$T(\varphi)$: warped brane tension

two free functions of 10d background geometry

"speed limit":
$$\boldsymbol{\gamma(\dot\varphi,\varphi) = \frac{1}{\sqrt{1-\dot\varphi^2/T(\varphi)}}}$$

$\gamma \approx 1$: standard regime
$\gamma \gg 1$: "ultrarelativistic" DBI regime

Silverstein and Tong (2003)
Alishahiha, Silverstein and Tong (2004)

DBI Langevin Equation and Perturbative Solution

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- ansatz as before $\varphi(t) = \varphi_{\rm cl}(t) + \delta\varphi_1(t) + \delta\varphi_2(t) + \ldots$, gives

$$\langle \delta\varphi_1^2 \rangle = \frac{\kappa}{2}\left(\frac{H'}{2\pi\gamma}\right)^2 \int_{\varphi_{\rm cl}}^{\phi_{\rm in}} {\rm d}\phi \left(\frac{H\gamma}{H'}\right)^3$$

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- use these to calculate

$$P(\varphi, t) = \frac{1}{\sqrt{2\pi\,\langle \delta\varphi_1^2 \rangle}} \exp\left[-\frac{(\varphi - \langle\varphi\rangle)^2}{2\,\langle \delta\varphi_1^2 \rangle}\right]$$

with $\langle\varphi\rangle = \varphi_{\rm cl} + \langle \delta\varphi_2 \rangle$

Example. . . and a Problem

$$V(\varphi) = V_0 + \frac{m^2}{2}\varphi^2$$
$$T(\varphi) = \frac{\varphi^4}{\lambda}$$

param's:
$m, \alpha = \frac{96\pi^2}{\kappa\lambda m^2}, \beta = \frac{V_0}{m^2 m_{\rm Pl}^2}$

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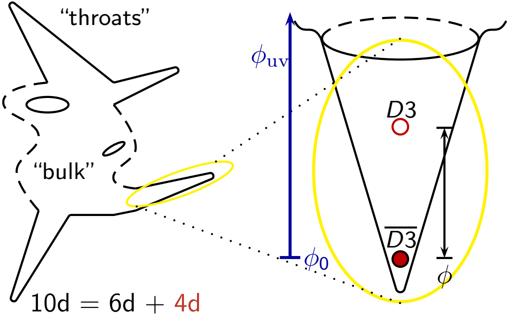

Klebanov & Strassler (2000)
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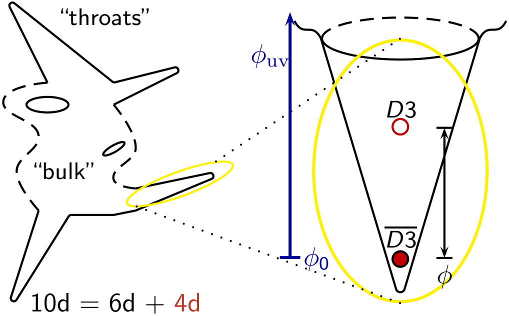

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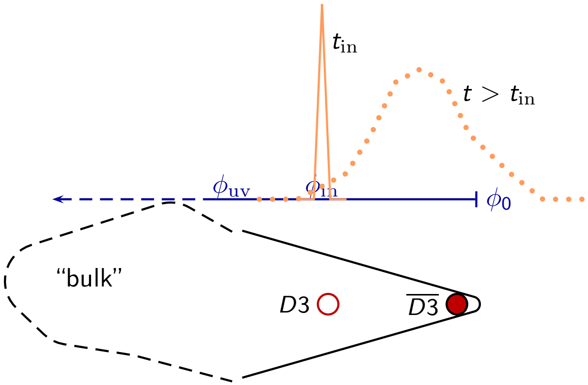
. . . and therefore limited field range!

"Out of Space"?

Problem: w/ stochastic corrections, probability of $\langle\varphi\rangle < \phi_0$ is $\neq 0$!



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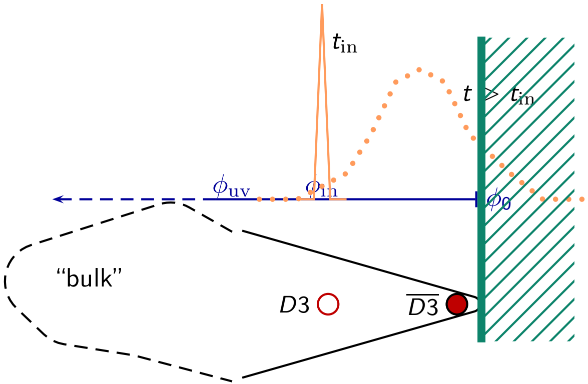
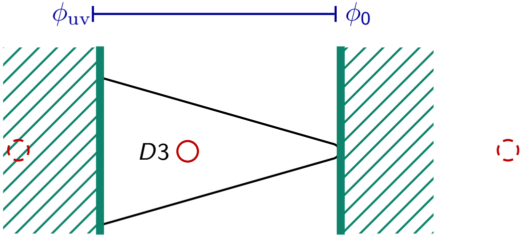


Solution: install absorbing wall(s) in field space and calculate resulting P

Boundary Conditions by Method of Images

$$\mathbf{P}(\varphi = \phi_0) = \mathbf{P}(\varphi = \phi_{\mathrm{uv}}) = 0$$



using $\phi_{\mathrm{mean}} \equiv \phi_{\mathrm{cl}} + \langle \delta\phi_2 \rangle = \langle \varphi \rangle_{\mathrm{w/o\ walls}}$:

$$P_{\mathrm{walls}}(\varphi) = P(\varphi - \phi_{\mathrm{mean}}) - P(\varphi - 2\phi_{\mathrm{uv}} + \phi_{\mathrm{mean}}) - P(\varphi - 2\phi_0 + \phi_{\mathrm{mean}})$$

Boundary Conditions (2)

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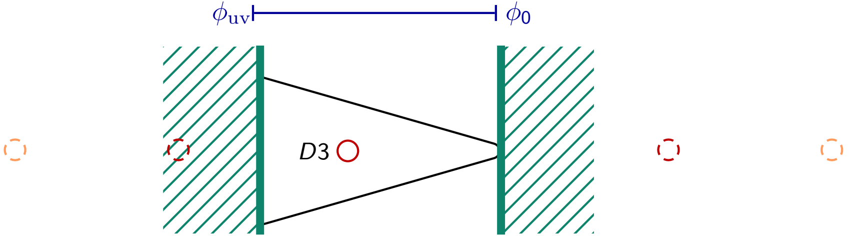

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$$P_{walls}(\varphi) = P(\varphi - \phi_{mean}) - P(\varphi - 2\phi_{uv} + \phi_{mean}) - P(\varphi - 2\phi_0 + \phi_{mean}) + P(\varphi - 2\phi_{uv} - \phi_{mean} + 2\phi_0) + P(\varphi - 2\phi_0 - \phi_{mean} + 2\phi_{uv})$$

Boundary Conditions (3)

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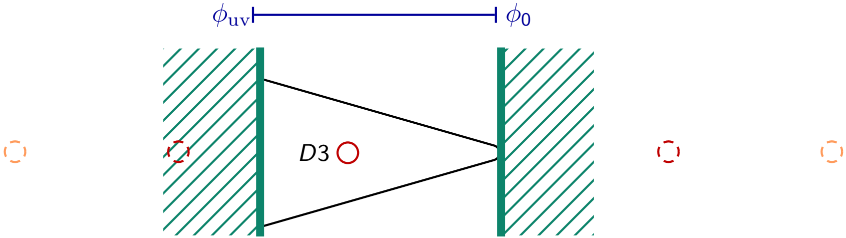


This process is repeated *ad infinitum*: $P_{\mathrm{walls}(\varphi)} = \frac{1}{\sqrt{2\pi\left\langle\delta\varphi_1^2\right\rangle}} \times$

$$\sum_{n=-\infty}^{\infty}\left(\exp\left\{-\frac{[\varphi-\phi_{\mathrm{mean}}+2n(\phi_{\mathrm{uv}}-\phi_0)]^2}{2\left\langle\delta\varphi_1^2\right\rangle}\right\} - \exp\left\{-\frac{[\varphi+\phi_{\mathrm{mean}}-2\phi_0+2n(\phi_{\mathrm{uv}}-\phi_0)]^2}{2\left\langle\delta\varphi_1^2\right\rangle}\right\}\right)$$

Mean Field Value between Absorbing Walls

Now use $P_{\rm walls}$ to calculate new $\langle\varphi\rangle = \frac{1}{N}\int_{\phi_0}^{\phi_{\rm uv}} \mathrm{d}\psi \, P_{\rm walls}(\psi, t)\,\psi$:

$$\int_{\phi_0}^{\phi_{\rm uv}} \mathrm{d}\psi \, P_{\rm walls}(\psi, t)\,\psi = \sum_{n=1}^{\infty} \frac{2}{n\pi} \exp\left[-\frac{n^2\pi^2 \left\langle \delta\varphi_1^2 \right\rangle}{2(\phi_{\rm uv} - \phi_0)^2}\right] \times \sin\left[\frac{n\pi(\phi_0 - \phi_{\rm mean})}{\phi_{\rm uv} - \phi_0}\right] \left[\phi_{\rm uv}\cos(n\pi) - \phi_0\right]$$

Oscillating behaviour!

Note:

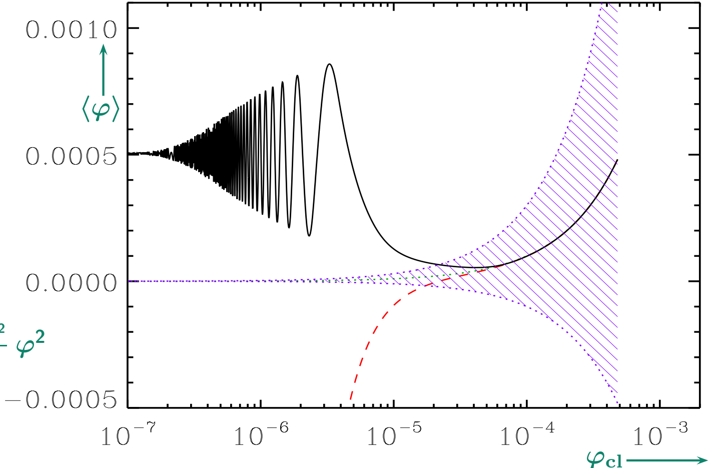
$P_{\rm walls}$ not normalized b/c of absorbing boundary conditions.

Need to calculate $N = \int_{\phi_0}^{\phi_{\rm uv}} \mathrm{d}\psi \, P_{\rm walls}(\psi, t)$ to find $\langle\varphi\rangle$.

applies to any stochastic inflaton between two absorbing walls

Example. . . and Problem Solved

$\alpha = 38, \beta = 3.7, m \simeq 2 \times 10^{-7}, \phi_{\rm in} = 10^{-3} m_{\rm Pl}$



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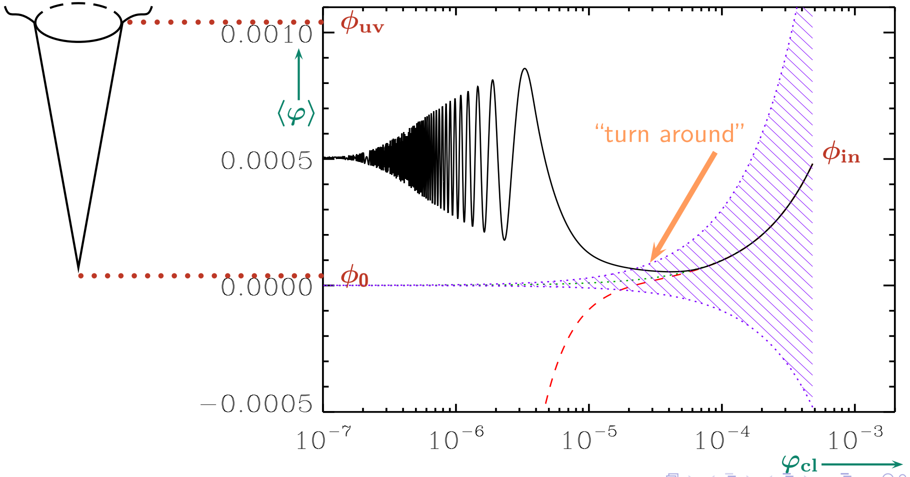

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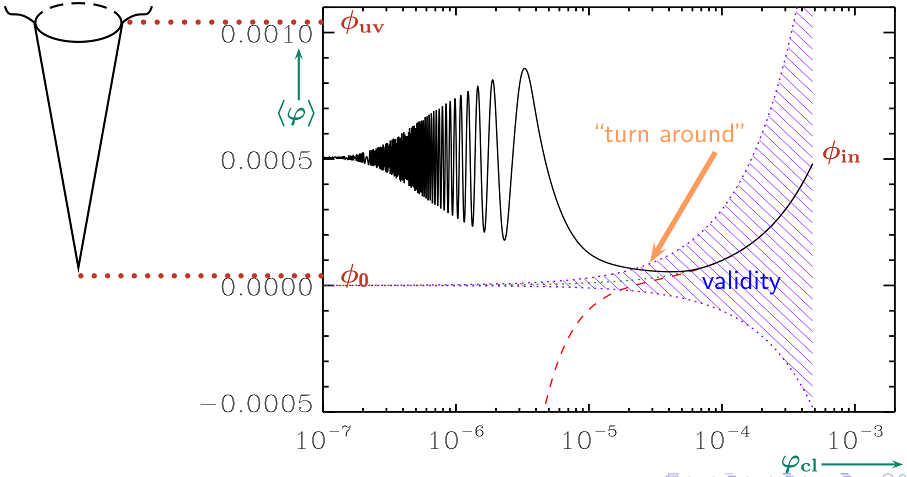

Conclusions

- In stochastic inflation, the smoothed field φ obeys a Langevin equation. This equation can be solved perturbatively in the noise.
- We generalized this approach to string-inspired DBI models.

But:
The stringy inflaton has a geometric interpretation.
Must be respected even by quantum corrections!

- Install walls in field space to find correct PDF's.
- This may affect the behaviour of the inflaton significantly!