Higher-Dimensional Kerr-Schild Spacetimes with (A)dS Background

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Kerr-Schild metric with (A)dS background

- Kerr-Schild metric $g_{ab} = \eta_{ab} - 2\mathcal{H}k_a k_b$
  - analytical tractable
  - KS class contains: Kerr, Myers-Perry, type N pp-waves ...

Generalized Kerr-Schild metric

$$g_{ab} = \bar{g}_{ab} - 2\mathcal{H}k_a k_b$$

- KS vector ***k*** is null with respect to the both metric
- inverse $g^{ab} = \bar{g}^{ab} + 2\mathcal{H}k^a k^b$
- canonical form of the background metric

$$\bar{g} = \Omega(-dt^2 + dx_1^2 + \ldots + dx_{n-1}^2)$$

where $\Omega_{dS} = \frac{(n-2)(n-1)}{2\Lambda t^2}$, $\quad \Omega_{AdS} = -\frac{(n-2)(n-1)}{2\Lambda {x_1}^2}$

Newman-Penrose formalism in higher dimensions

real null frame $\boldsymbol{n} \equiv \boldsymbol{m}^{(0)}$, $\boldsymbol{\ell} \equiv \boldsymbol{m}^{(1)}$, $\boldsymbol{m}^{(i)}$

$$l^a l_a = n^a n_a = l^a m_a^{(i)} = n^a m_a^{(i)} = 0$$

$$l^a n_a = 1\,, \qquad m^{(i)a} m_a^{(j)} = \delta_{ij}$$

- $i, j, \ldots$ range from 2 to $n-1$
  $a, b, \ldots$ from 0 to $n-1$
- metric in the null frame

$$g_{ab} = 2n_{(a}l_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}$$

- we identify $\boldsymbol{k}$ with $\boldsymbol{\ell}$

Newman-Penrose formalism in higher dimensions

Ricci rotation coefficients

$$l_{a;b} = L_{cd}\, m_a^{(c)} m_b^{(d)} \,, \quad n_{a;b} = N_{cd}\, m_a^{(c)} m_b^{(d)} \,, \quad m_{a;b}^{(i)} = \overset{i}{M}_{cd}\, m_a^{(c)} m_b^{(d)}$$

- if $\boldsymbol{k}$ is geodetic ($L_{i0} = L_{10} = 0$)
  - decomposition of L_{ij}:

$$S_{ij} \equiv L_{(ij)} = \sigma_{ij} + \theta\delta_{ij} \,, \quad A_{ij} \equiv L_{[ij]}$$

  - θ, σ_{ij} and A_{ij} are the *expansion*, *shear* and *twist* with corresponding scalars

$$\sigma^2 = \sigma_{ij}\sigma_{ij} \,, \quad \omega^2 = A_{ij}A_{ij}$$

Directional derivatives along the frame vectors

$$\mathrm{D} \equiv k^a \nabla_a \,, \quad \Delta \equiv n^a \nabla_a \,, \quad \delta_i \equiv m_{(i)}^a \nabla_a \,.$$

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

- Weyl tensor frame components sorted by boost weight

frame components				boost weight
C_{0i0j}				2
C_{010i}	C_{0ijk}			1
C_{0101}	C_{01ij}	C_{0i1j}	C_{ijkl}	0
C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

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C_{1i1j}				-2

type G

Algebraic classification in higher dimensions

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- boost of the frame

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C_{1i1j}				-2

type I

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

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C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type I_i

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

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C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type II

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

- Weyl tensor frame components sorted by boost weight

frame components				boost weight
C_{0i0j}				2
C_{010i}	C_{0ijk}			1
C_{0101}	C_{01ij}	C_{0i1j}	C_{ijkl}	0
C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type II_i

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

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C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type III

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

- Weyl tensor frame components sorted by boost weight

frame components				boost weight
C_{0i0j}				2
C_{010i}	C_{0ijk}			1
C_{0101}	C_{01ij}	C_{0i1j}	C_{ijkl}	0
C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type III_i

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

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C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type N

Algebraic classification in higher dimensions

- boost weight w: $\hat{q} = \lambda^w q$
- boost of the frame

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}$$

- Weyl tensor frame components sorted by boost weight

frame components				boost weight
C_{0i0j}				2
C_{010i}	C_{0ijk}			1
C_{0101}	C_{01ij}	C_{0i1j}	C_{ijkl}	0
C_{101i}	C_{1ijk}			-1
C_{1i1j}				-2

type D

KS vector $\boldsymbol{k}$ is geodetic iff T_{00} vanishes

- Ricci component $R_{00} = R_{ab} k^a k^b$

$$R_{00} = 2\mathcal{H} k_{c;a} k^a k^c{}_{;b} k^b - \frac{1}{2}(n-2)\left(\frac{\Omega_{,ab}}{\Omega} - \frac{3}{2}\frac{\Omega_{,a}\Omega_{,b}}{\Omega^2}\right) k^a k^b$$

Proposition

The vector $\boldsymbol{k}$ in the generalized Kerr-Schild metric is geodetic iff the component of the energy-momentum tensor $T_{00} = T_{ab} k^a k^b = 0$ vanishes.

$\boldsymbol{k}$ geodetic for

- Einstein spaces
- spacetimes with aligned matter fields (Maxwell field $F_{ab} k^a \propto k_b$, aligned pure radiation $T_{ab} \propto k_a k_b$)

Geodetic and optical properties in both metrics

- ***k*** is geodetic in the full metric
  iff it is geodetic in the background metric

$$k_{a;b}k^b = k_{a,b}k^b = k_{a\bar{;}b}k^b \,, \quad k^a{}_{;b}k^b = k^a{}_{,b}k^b + \frac{\Omega_{,b}}{\Omega}k^a k^b = k^a{}_{\bar{;}b}k^b$$

null frame in the background

$$\bar{g}_{ab} = 2k_{(a}\tilde{n}_{b)} + \delta_{ij}m^{(i)}_a m^{(j)}_b \,, \qquad \tilde{n}_a = n_a + \mathcal{H}k_a$$

- optical matrices in the full and background metric are equal

$$L_{ij} \equiv k_{a;b}m^{(i)a}m^{(j)b} = k_{a\bar{;}b}m^{(i)a}m^{(j)b} \equiv \tilde{L}_{ij}$$

Ricci tensor

let's assume geodetic $\boldsymbol{k}$

Ricci tensor

$$R_{ab} = (\mathcal{H}k_a k_b)_{;cd}\, g^{cd} - (\mathcal{H}k^s k_a)_{;bs} - (\mathcal{H}k^s k_b)_{;as} + \frac{2\Lambda}{n-2}\bar{g}_{ab} - 2\mathcal{H}\left(\mathrm{D}^2\mathcal{H} + L_{ii}\mathrm{D}\mathcal{H} + 2\mathcal{H}\omega^2\right) k_a k_b$$

- $\boldsymbol{k}$ is eigenvector of Ricci
  $$R_{ab}k^b = -\left[\mathrm{D}^2\mathcal{H} + (n-2)\theta\mathrm{D}\mathcal{H} + 2\mathcal{H}\omega^2 - \frac{2\Lambda}{n-2}\right] k_a$$
- frame components
  $$R_{00} = 0$$
  $$R_{01} = -\mathrm{D}^2\mathcal{H} - (n-2)\theta\mathrm{D}\mathcal{H} - 2\,\mathcal{H}\omega^2 + \frac{2\Lambda}{n-2}$$
  $$R_{ij} = 2\mathcal{H}L_{ik}L_{jk} - 2\left(\mathrm{D}\mathcal{H} + (n-2)\theta\mathcal{H}\right)S_{ij} + \frac{2\Lambda}{n-2}\delta_{ij}$$

KS spacetime with geodetic $\boldsymbol{k}$ is at least of type II

- positive boost weight frame components of Weyl vanish
  $C_{0i0j} = 0, \qquad C_{010i} = 0, \qquad C_{0ijk} = 0$

Proposition

Generalized Kerr-Schild spacetime with geodetic vector $\boldsymbol{k}$ is algebraically special with $\boldsymbol{k}$ being the multiple WAND.

- [Pravda Pravdová Ortaggio 2007]: static (or stationary with “reflection symmetry”) expanding spacetimes are of types G, I_i, D or O

Corollary

Static (or stationary with “ref. sym.”) generalized Kerr-Schild spacetimes with geodetic vector $\boldsymbol{k}$ are of type D or O.

Vacuum equations

from now on we restrict us to Einstein spaces

- n-dimensional EFEs:

$$R_{ab} = \frac{2}{n-2}\Lambda g_{ab}$$

- $R_{ij} = \frac{2}{n-2}\Lambda\delta_{ij}$ component can be written as

$$(n-2)\theta(\mathrm{D}\log\mathcal{H}) = \sigma^2 + \omega^2 - (n-2)(n-3)\theta^2$$

Nonexpanding Einstein KS spaces belong to Kundt N

- vanishing expansion $\theta = 0$ implies vanishing shear $\sigma = 0$ and twist $\omega = 0$, thus $L_{ij} = 0$
- substituting EFEs to Weyl: boost weight 0 and -1 components vanish

Proposition

Einstein Kerr-Schild spacetimes with non-expanding KS congruence $\boldsymbol{k}$ are of type N with $\boldsymbol{k}$ being the multiple WAND. Twist and shear of the KS congruence $\boldsymbol{k}$ necessarily vanish and these solutions thus belong to the class of Einstein type N Kundt spacetimes.

Optical constraint

- equation $R_{ij} = \frac{2}{n-2}\Lambda\delta_{ij}$ now nontrivial

Optical constraint

$$L_{ik}L_{jk} = \frac{L_{ik}L_{ik}}{(n-2)\theta}S_{ij}$$

- L_{ij} normal matrix: $[L, L^T] = 0$
- in appropriate frame L_{ij} has block-diagonal form consisting of 2x2 blocks: $\begin{pmatrix} \mathcal{S} & \mathcal{A} \\ -\mathcal{A} & \mathcal{S} \end{pmatrix}$
- types III and N not compatible with expanding Einstein KS

Proposition

Einstein Kerr-Schild spacetimes with expanding KS congruence k are of Weyl types II or D or conformally flat

r-dependece of optical matrix L_{ij}

- block-diagonal form + Sachs equation ($\mathrm{D}L_{ik} = -L_{ik}L_{kj}$):

$$L_{ij} = \begin{pmatrix} \boxed{\mathcal{L}_{(1)}} & & & \\ & \ddots & & \\ & & \boxed{\mathcal{L}_{(p)}} & \\ & & & \boxed{\tilde{\mathcal{L}}} \end{pmatrix}$$

$$\mathcal{L}_{(\mu)} = \begin{pmatrix} s_{(2\mu)} & A_{2\mu,2\mu+1} \\ -A_{2\mu,2\mu+1} & s_{(2\mu)} \end{pmatrix} \qquad (\mu = 1, \dots, p),$$

$$s_{(2\mu)} = \frac{r}{r^2 + (a^0_{(2\mu)})^2}, \qquad A_{2\mu,2\mu+1} = \frac{a^0_{(2\mu)}}{r^2 + (a^0_{(2\mu)})^2},$$

$$\tilde{\mathcal{L}} = \frac{1}{r}\mathrm{diag}(\underbrace{1,\dots,1}_{(m-2p)},\underbrace{0,\dots,0}_{(n-2-m)})$$

Singularities

integrating $R_{ij} = \frac{2}{n-2}\Lambda\delta_{ij}$: $\mathcal{H} = \frac{\mathcal{H}_0}{r^{m-2p-1}} \prod_{\mu=1}^{p} \frac{1}{r^2+(a^0_{(2\mu)})^2}$

- divergences at $r = 0$
  1. $2p \neq m$, $2p \neq m-1$
  2. $2p = m$ (m even), $2p = m-1$ (m odd) if some of $a^0_{(2\mu)}$ vanish somewhere in spacetime

Kretschmann scalar

$$R_{abcd}R^{abcd} = 4\Phi^2 + 8\Phi^S_{ij}\Phi^S_{ij} - 24\Phi^A_{ij}\Phi^A_{ij} + C_{ijkl}C_{ijkl} + \frac{8n}{(n-1)(n-2)^2}\Lambda^2$$

- where $\Phi_{ij} \equiv C_{0i1j}$, then $\Phi = C_{0101}$, $\Phi^S_{ij} = -\frac{1}{2}C_{ikjk}$, $\Phi^A_{ij} = \frac{1}{2}C_{01ij}$

Kerr-de Sitter

Kerr-(A)dS in KS form [Gibbons et al. (2004)]

$$g_{ab} = \bar{g}_{ab} + \frac{2M}{\rho^2} k_a k_b$$

- in case of 5D

$$\bar{g} = -\frac{(1-\lambda r^2)\Delta}{(1+\lambda a^2)(1+\lambda b^2)}\, dt^2 + \frac{r^2\rho^2}{(1-\lambda r^2)(r^2+a^2)(r^2+b^2)}\, dr^2$$
$$+\frac{\rho^2}{\Delta}\, d\theta^2 + \frac{(r^2+a^2)\sin^2\theta}{1+\lambda a^2}\, d\phi^2 + \frac{(r^2+b^2)\cos^2\theta}{1+\lambda b^2}\, d\psi^2$$

$$k = \frac{\Delta}{(1+\lambda a^2)(1+\lambda b^2)}\, dt + \frac{r^2\rho^2}{(1-\lambda r^2)(r^2+a^2)(r^2+b^2)}\, dr$$
$$-\frac{a\sin^2\theta}{1+\lambda a^2}\, d\phi - \frac{b\cos^2\theta}{1+\lambda b^2}\, d\psi$$

where $\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \quad \Delta = 1 + \lambda a^2\cos^2\theta + \lambda b^2\sin^2\theta$

Comparison of 5D Kerr-de Sitter with general results

5D Kerr-de Sitter:	general KS ($n=5$, $m=3$, $p=1$):
$L = \begin{pmatrix} \frac{r}{\rho^2} & \frac{\sqrt{\rho^2-r^2}}{\rho^2} & 0 \\ -\frac{\sqrt{\rho^2-r^2}}{\rho^2} & \frac{r}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{r} \end{pmatrix}$	$L = \begin{pmatrix} s_{(2)} & A_{2,3} & 0 \\ -A_{2,3} & s_{(2)} & 0 \\ 0 & 0 & \frac{1}{r} \end{pmatrix}$
$\mathcal{H} = -M \frac{1}{r^2+a^2 cos^2\theta+b^2\sin^2\theta}$	$\mathcal{H} = \mathcal{H}_0 \frac{1}{r^2+(a^0_{(2)})^2}$
$s_{(2)} = \frac{r}{r^2+a^2\cos^2\theta+b^2\sin^2\theta}$	$s_{(2)} = \frac{r}{r^2+(a^0_{(2)})^2}$
$A_{(2,1)} = \frac{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}{r^2+a^2\cos^2\theta+b^2\sin^2\theta}$	$A_{(2,1)} = \frac{a^0_{(2)}}{r^2+(a^0_{(2)})^2}$

- $\mathcal{H}_0 = -M\,, \qquad a^0_{(2)} = a^2\cos^2\theta + b^2\sin^2\theta$
  - $a \neq 0$, $b \neq 0$: no singularity
  - $a \neq 0$, $b = 0$: singularity if $\theta = \frac{\pi}{2}$
  - $a = b = 0$ (de Sitter-Schwarzschild-Tangherlini) p=0: singularity

Conclusions

- GKS are of type II or more special
- KS vector $\boldsymbol{k}$ is geodetic iff $T_{00} = 0$
- non-expanding Einstein GKS belong to type N Einstein Kundt class (equivalence at least in Ricci-flat case)
- expanding Einstein GKS
  - are of type II, D or O
  - r-dependence of L_{ij} and Weyl bw 0 components explicitly determined
  - condition for presence of a singularity at $r = 0$