Computational Methods in General Relativity

From exact tensor computations to critical phenomena in gravitational collapse

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  - Finite precision numbers: information loss in basic operations. Example:

$$x^2 - 2(1 \pm \epsilon)x + (1 \pm \epsilon) = 0, \qquad \Rightarrow \qquad x = 1 \pm \sqrt{3\epsilon}$$

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Kranc (Husa, Hinder, Lechner)
RNPL (Marsa, Choptuik)
Cactus, Einstein Toolkit (AEI & LSU)

xAct

- Manifolds, vector bundles, tensors, connections, metrics.
- Frames, charts.
- Abstract indices vs. frame indices. 100 indices in seconds.
- Modules for spinors (García-Parrado, M-G), perturbation theory (Brizuela, M-G, Mena Marugán), Riemann scalars (M-G, Yllanes, Portugal), ...
- Some applications:
  - Super-energy tensors (García-Parrado)
  - Cosmological perturbation theory (Pitrou)
  - Hyperbolicity of Einstein eqs (Gundlach, M-G)
  - PostNewtonian computations (Faye et al)
  - Heat-kernel expansions (Wardell et al)
  - Geometric invariants (Backdahl, Valiente Kroon) [prev talk]
  - IVP on light-cones (Choquet-Bruhat, Chruściel, M-G)
  - QFT, string theory, ...
- `http://www.xAct.es`
  `http://groups.google.com/group/xAct`

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Motivation from mathematics

- *Nonlinear stability* of Minkowski in vacuum GR. (Ex: $\dot{y}(t) = y(t)^2$).
- *Cosmic censorship*: is it possible to form a naked (visible to far observers) singularity starting from smooth initial conditions in a self-gravitating system which is regular without gravity?

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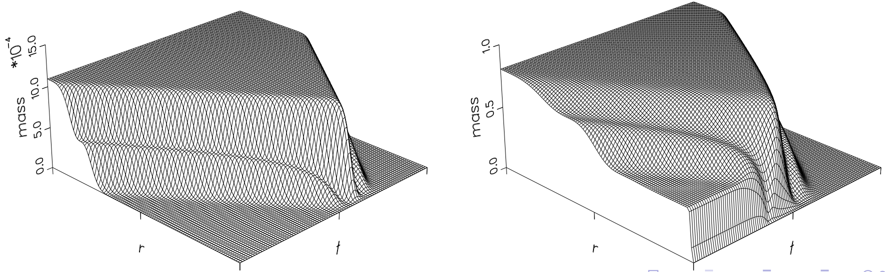
  - Results (CMP'86):
    - Small finite data $\Rightarrow$ Minkowski is stable.
    - Large data $\Rightarrow$ Schwarzschild end state.



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- Goldwirth and Piran, PRD'87:
  We present a numerical study of the gravitational collapse of a massless scalar field. We calculate the future evolution of new initial data, suggested by Christodoulou, and we show that in spite of the original expectations these data lead only to singularities engulfed by an event horizon.

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- Choptuik, Goldwirth and Piran CQG'92: compare codes

  [CA≡ Cauchy (Choptuik's code). CH≡ Characteristic (GP's code).]

  ... although the levels of error in the CA and CH results at a given resolution were quite comparable at early retarded times (...), the CA values were significantly more accurate than the CH data once the pulse of scalar field had reached $r = 0$.

Choptuik's setup

- The system:

$$ds^2 = -\alpha^2(t,r)dt^2 + a^2(t,r)dr^2 + r^2 d\Omega^2, \qquad \Phi \equiv \phi', \quad \Pi \equiv a\dot{\phi}/\alpha$$

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)', \quad \dot{\Pi} = \frac{1}{r^2}\left(r^2\frac{\alpha}{a}\Phi\right)', \qquad \frac{\alpha'}{\alpha} = \frac{a'}{a} + \frac{a^2-1}{r} = 2\pi r(\Pi^2 + \Phi^2).$$

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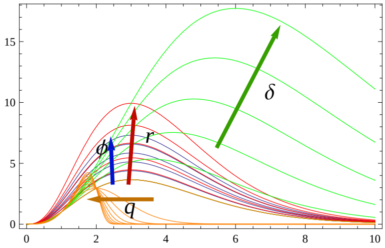
Example (pure ingoing):

$$\phi(0,r) = \phi_0\, r^3 \exp\left(-[(r-r_0)/\delta]^q\right)$$

$$p = \phi_0, r_0, \delta, q$$



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Comment: Self-similarity is dynamically found, but in a more general (DSS) form!

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- Proca, Dirac, sigma fields, ..., Vlasov(?)
- With/without mass, charge, conformal couplings, ...
- Different equations of state for fluids.
- Other dimensions.

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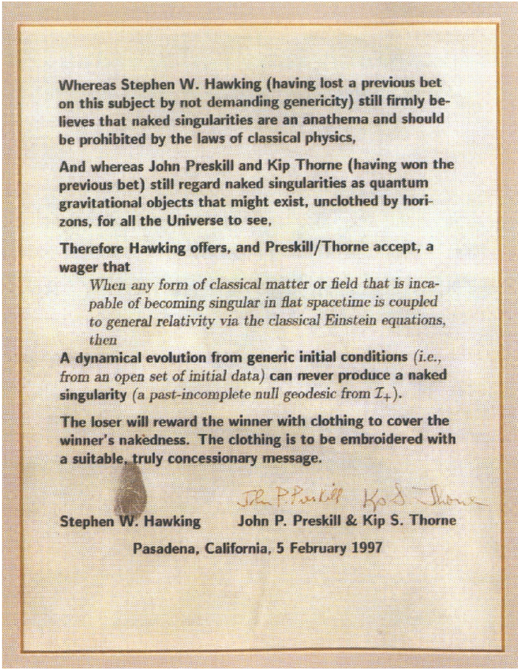
- New bet!



Whereas Stephen W. Hawking (having lost a previous bet on this subject by not demanding genericity) still firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,

And whereas John Preskill and Kip Thorne (having won the previous bet) still regard naked singularities as quantum gravitational objects that might exist, unclothed by horizons, for all the Universe to see,

Therefore Hawking offers, and Preskill/Thorne accept, a wager that

When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, then

A dynamical evolution from generic initial conditions *(i.e., from an open set of initial data)* **can never produce a naked singularity** *(a past-incomplete null geodesic from* $\mathcal{I}_+$*).*

The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable, truly concessionary message.

Stephen W. Hawking — **John P. Preskill & Kip S. Thorne**

Pasadena, California, 5 February 1997

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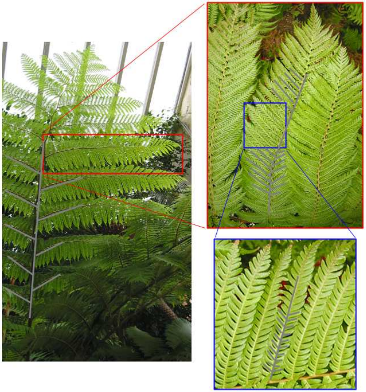

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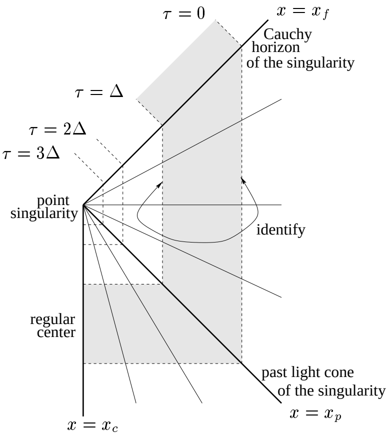

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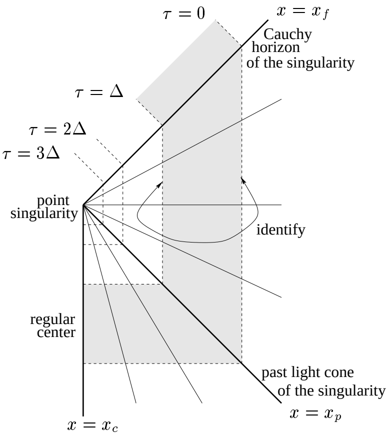

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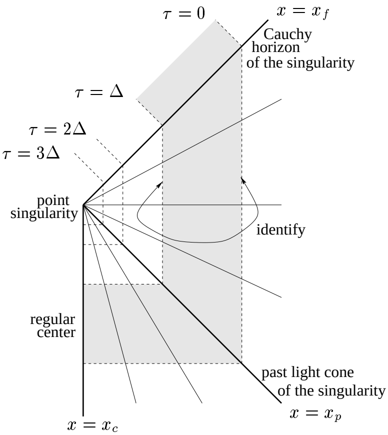
  DSS: also periodic in τ, period Δ.



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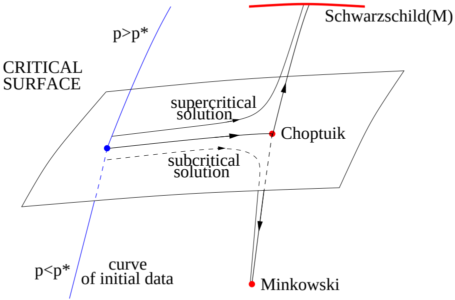

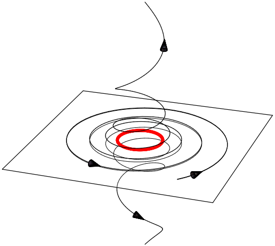

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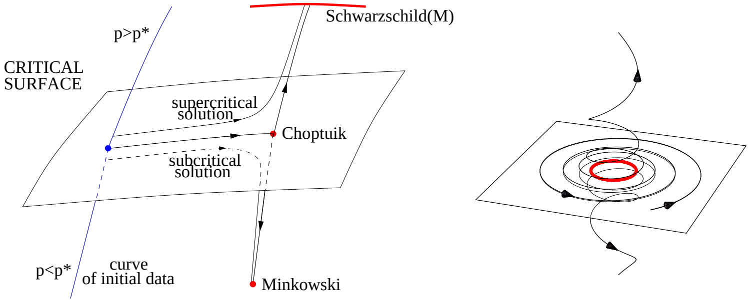

- Open questions:
  - Which functional space? Asymptotic properties of the spacetimes.
  - Which foliations? Which coordinates?
  - Meaning of "attraction"?

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- Attraction $\Rightarrow$ Forget initial details $\Rightarrow$ Highly symmetric solutions:
  - Spherical or axisymmetric
  - Static ("type I") or self-similar ("type II"). Both continuous or discrete.

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- Garfinkle, Gundlach & M-G PRD'99 : Conjectured scaling law for angular momentum, exponent 0.762(2).

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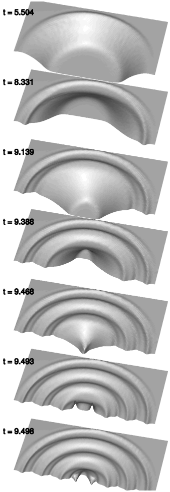

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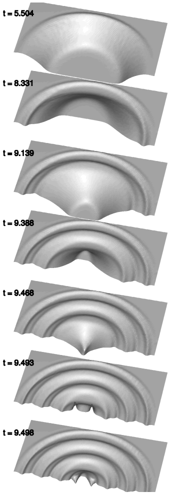

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- Choptuik et al PRL'04: ansatz $\phi(t, \rho, z, \phi) = e^{im\phi}\psi(t, \rho, z)$. DSS criticality. Isolated m sectors. Which unstable?

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  - $k < 1/9$ (analytical): $l = 1$ axial unstable (ballerina effect).
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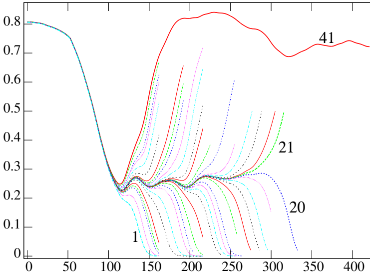
Non-linear results:

- Jin & Suen PRL'07: BH threshold in neutron stars head-on collision.
  - Signs of type I criticality.
  - Critical solution: oscillating spherical neutron star, probably a perturbed unstable TOV star (Noble & Choptuik '08).



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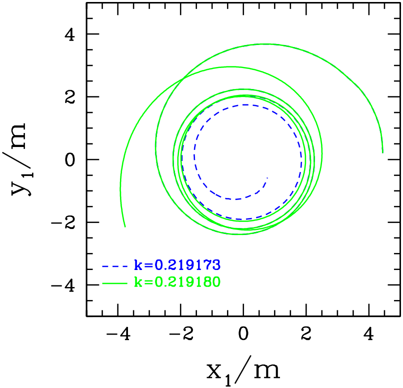

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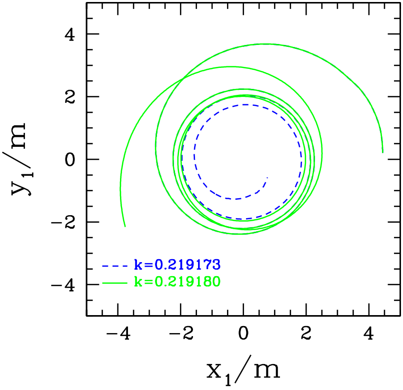
- N circular orbits before merging or dispersing.
- $e^N \propto (p - p^*)^{-\gamma}$, with $\gamma \approx 0.31 - 0.38$
- 1.5% total energy radiated per orbit.
- max N limited by kinetic energy available.
- Self-similar criticality for zero mass BHs?



4. Global structures for a self-similar spacetime

- Recall structure $e^{-2\tau} g_{\mu\nu}(x)$. Central singularity at $\tau = \infty$.
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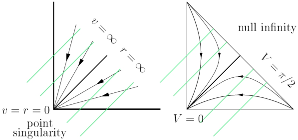

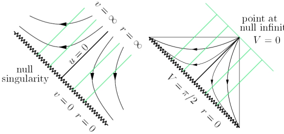

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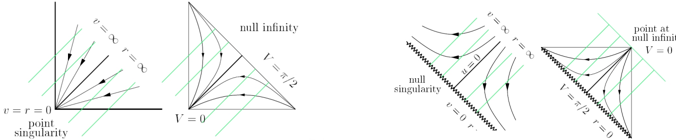
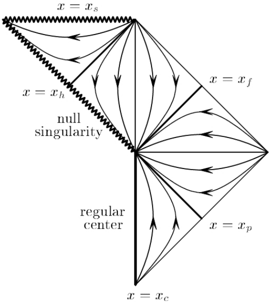


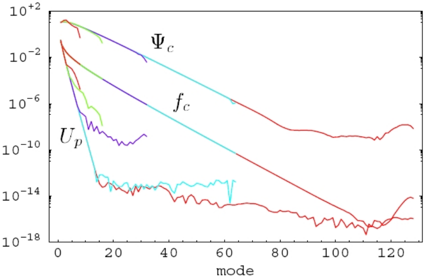
- Example:

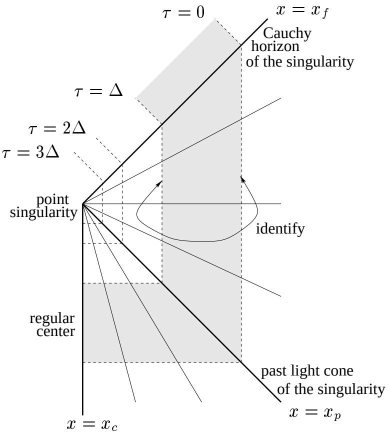


5. High precision numerical Choptuik spacetime

M-G & Gundlach PRD'03

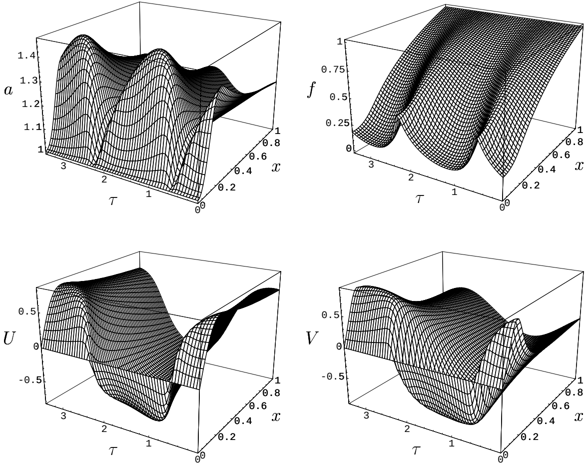
- Three regions
- Psedospectral code. Fourier in τ; 4^{th} order FD in x.





Inner patch: Impose DSS and regularity at centre and past light cone.

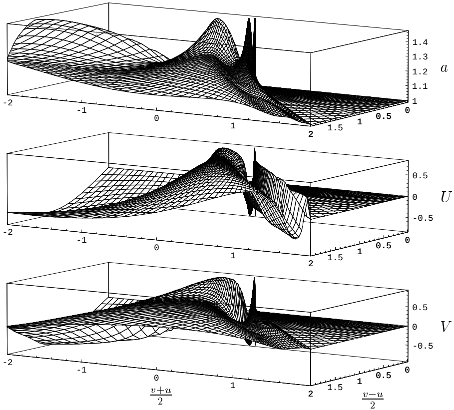
$$\Delta = 3.445\,452\,402(3)$$



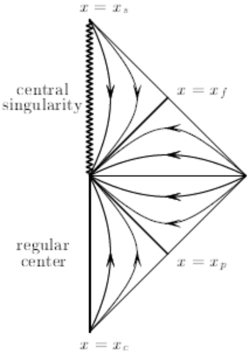
Confirmed by Grandclement'09 (*kaddath*).

6. Global structure of the Choptuik spacetime

- Oscillations pile up at the Cauchy Horizon, but decay.
- Curvature is continuous but non-differentiable. Continuation not unique: one free function (radiation from the singularity).
- Unique DSS continuation with regular center (nearly flat):



All other continuations produce a negative mass singularity at the centre, with no new self-similarity horizon:



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Bizoń et al PRD'05, PRL'05, PRL'06

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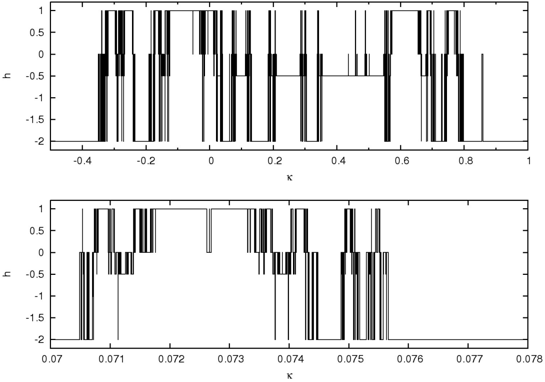
- κ-family of ICs. Possible end-states $h = 1, 1/2, -2$ or 0 (unknown).



Summary

Linking MathRel and NumRel?

Critical phenomena in gravitational collapse

- Christodoulou
- Choptuik
- The current model
- Interesting results

Conclusions and open questions

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- Dynamical understanding of the process missing. Why DSS?
- What happens outside spherical symmetry? Angular momentum?
- Relation with Christodoulou '94 '99?
- Show existence of the Choptuik spacetime.
- Can we approximate critical exponents analytically? Holography?

Gundlach & M-G, Living Reviews Relativity 2007, updated 2010.