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From emission to inertial coordinates: an analytical approach

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- B. Coll et al. *Positioning systems in Minkowski spacetime: from emission to inertial coordinates* CQG **27** (2010) 06501

Relativistic positioning systems

- Let us consider a relativistic positioning system in Minkowski space-time, i.e. four emitters γ_A broadcasting their proper time τ^A $(A = 1, 2, 3, 4)$.
- Let P be an event of the *emission region* $\mathcal{R}$, that is, a user at P receives the four broadcast times $\{\tau^A\}$ (emission coordinates of P).

The vector $m_A = x - \gamma_A$ is :

- null: $m_A^2 = 0$,
- future-pointing: $\epsilon u \cdot m_A < 0$.

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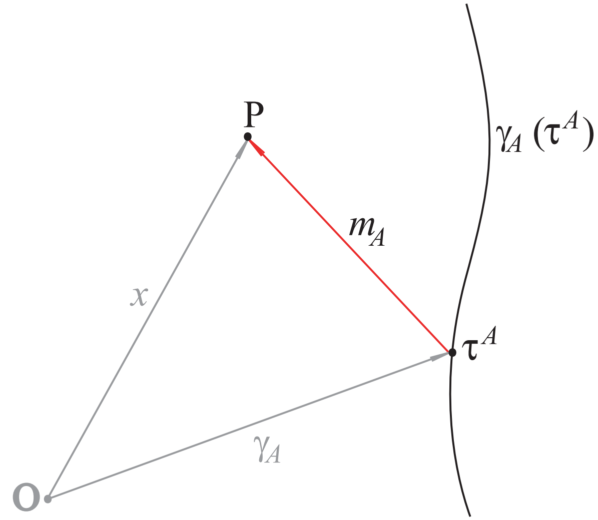

- m_A gives the trajectory followed by the electromagnetic signal from the emitter $\gamma_A(\tau^A)$ to the reception event $P \in \mathcal{R}$.

◇ The transformation $x^\alpha = \kappa^\alpha(\tau^A)$ from emission $\{\tau^A\}$ to inertial $\{x^\alpha\}$ coordinates is the solution of the emission equations:

$$(x - \gamma_A) \cdot (x - \gamma_A) = 0\,, \qquad \epsilon\, u \cdot (x - \gamma_A) < 0\,, \qquad A = 1, 2, 3, 4\,,$$

where $x \equiv (x^\alpha)$, $\gamma_A \equiv \gamma_A(\tau^A)$,

2ϵ is the metric signature, and

u is a future-pointing time-like vector.

The emission equations say that the vectors $m_A = x - \gamma_A$ are:

- null: $m_A^2 = 0$,
- future-pointing: $\epsilon u \cdot m_A < 0$.

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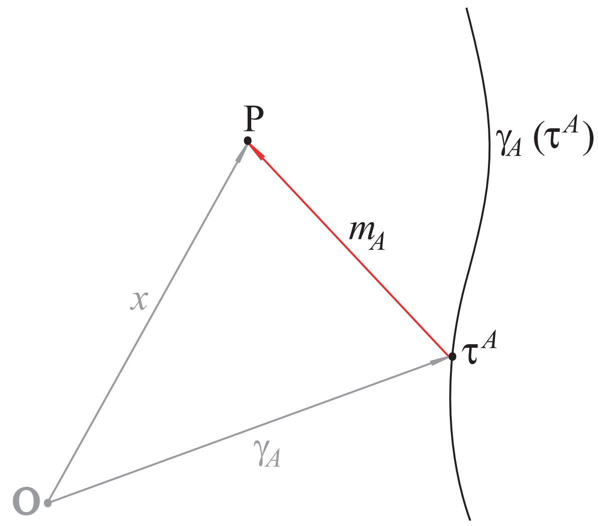


The configuration vector χ

- The emission data $\{\tau^A\}$ received at P are the emission coordinates of the event $P \in \mathcal{R}$ and were broadcast at the emission events $\{\gamma_A(\tau^A)\}$:

$$\{\tau^1, \tau^2, \tau^3, \tau^4\} \qquad \hookleftarrow \qquad \{\gamma_1(\tau^1), \gamma_2(\tau^2), \gamma_3(\tau^3), \gamma_4(\tau^4)\}$$

  The hyperplane generated by the four emission events $\{\gamma_A(\tau^A)\}$ is called the configuration of the emitters for P.

- The configuration vector

$$\chi$$

is orthogonal to this hyperplane.

- Emission region, $\mathcal{R} \subseteq \mathcal{M}^4$: space-time region reached by the signals.
- Emission function:

$$\Theta : \mathcal{R} \longrightarrow \mathcal{T} \equiv \overset{4}{\times} \{\tau\} \approx \mathbb{R}^4 \qquad\qquad \Theta : x \longmapsto (\tau^A) = \Theta(x)$$

- Emission coordinate region, $\mathcal{C} \subset \mathcal{R}$: where Θ is invertible: $\kappa = \Theta^{-1}\,, \quad x^\alpha = \kappa(\tau^A)$
- Coordinate condition:

$$d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4 \neq 0 \quad \Longleftrightarrow \quad j_\Theta(x) \neq 0 \quad \Longrightarrow \quad \chi \neq 0$$

- Zero Jacobian hypersurface: $\mathcal{J} \equiv \{x \,|\, j_\Theta(x) = 0\}, \quad \mathcal{R} = \mathcal{C} \cup \mathcal{J}$.

The coordinate transformation

Let us suppose that the world-lines $\gamma_A(\tau^A)$ of the emitters in an inertial system $\{x^\alpha\}$ are known

In all the emission coordinate region $\mathcal{C}$ the coordinate transformation $x = \kappa(\tau^A)$ is given by:

$$x = \gamma_4 + y_* - \frac{y_*^2\, \chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}}$$

$$y_*, \qquad \chi, \qquad \hat{\epsilon}$$

◇ Covariant solution: $x = f(\gamma_A) = f(\gamma_A(\tau^A)) = \kappa(\tau^A)$

The coordinate transformation

- Quantities y_*, χ are both computable from $\gamma_A(\tau^A)$.

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi) H, \qquad H \equiv *(\Omega_1\, e_2 \wedge e_3 + \Omega_2\, e_3 \wedge e_1 + \Omega_3\, e_1 \wedge e_2)$$

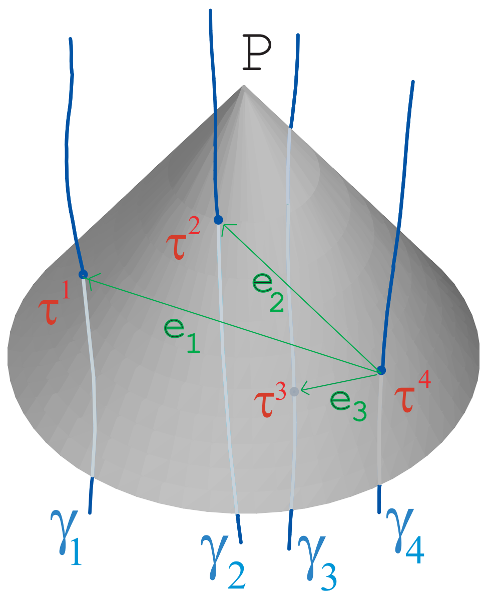
ξ being any vector transversal to the configuration, $\xi \cdot \chi \neq 0$, and

$$e_a = \gamma_a - \gamma_4 \quad (a = 1, 2, 3)$$

$$\Omega_a = \tfrac{1}{2}(e_a)^2$$

$$\chi \equiv *(e_1 \wedge e_2 \wedge e_3)$$

$\chi \equiv$ configuration vector



The coordinate transformation

- Quantity $\hat{\epsilon}$ is the orientation of the positioning system with respect to the event that receives the data $\{\tau^A\}$.

$$\hat{\epsilon} \equiv \textit{sgn} * (m_1 \wedge m_2 \wedge m_3 \wedge m_4),$$

$$m_A \equiv x - \gamma_A(\tau^A), \quad (A = 1, ..., 4),$$

- Problem: To obtain x from (1) one needs to determine the orientation $\hat{\epsilon}$, which involves the unknown x.

- Therefore, in order to show that the formula:

$$x = \gamma_4 + y_* - \frac{y_*^2 \chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}} \tag{1}$$

  does not chase its own tail, one must be able to determine the orientation $\hat{\epsilon}$ at x by using a procedure that does not involve the previous knowledge of x.

Problem

- To determine the orientation

$$\hat{\epsilon}$$

of the relativistic positioning system.

- Next, we study the region where $\hat{\epsilon}$ is computable by the positioning data (the central region of the positioning system) which does not cover the whole emission coordinate region $\mathcal{C}$.

Emission coordinate region: $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t$

Space-like configuration region:

$$\mathcal{C}_s \equiv \{x \in \mathcal{C} \,|\, \epsilon\, \chi^2 < 0\}$$

Light-like configuration region:

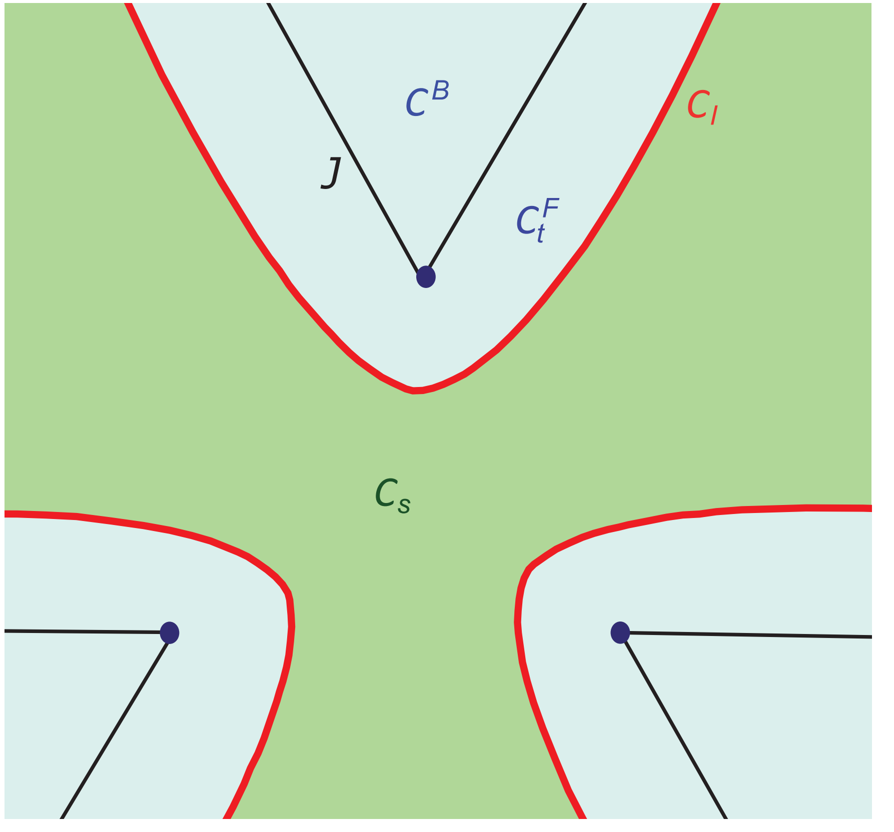
$$\mathcal{C}_\ell \equiv \{x \in \mathcal{C} |\, \chi^2 = 0\}$$

Time-like configuration region:

$$\mathcal{C}_t \equiv \{x \in \mathcal{C} \,|\, \epsilon\, \chi^2 > 0\}$$

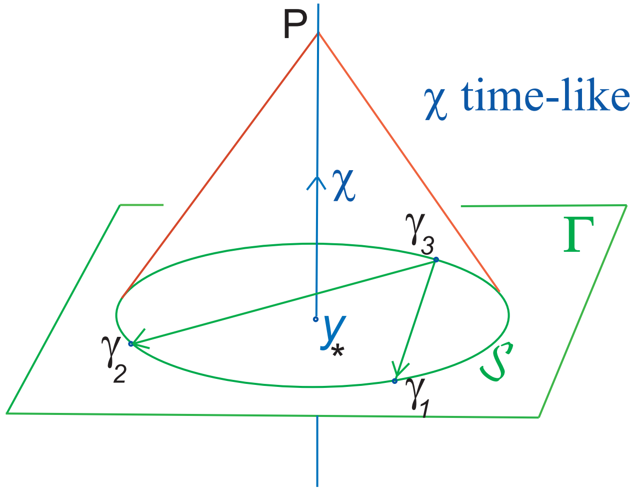
Central region:

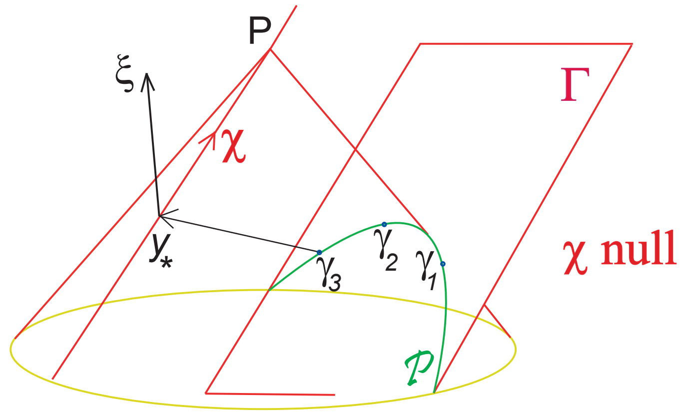
$$\mathcal{C}^C = \mathcal{C}_s \cup \mathcal{C}_\ell$$

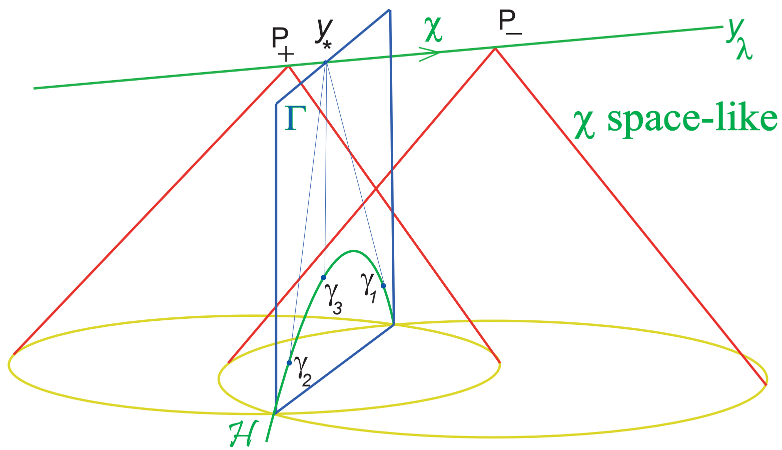


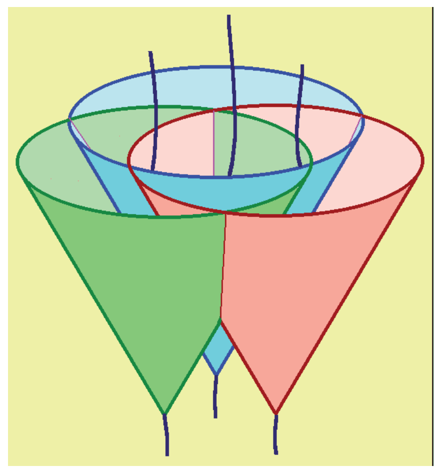
3D static situation

Non-uniqueness of emission solutions









Emission regions and coordinate domains

Emission coordinate region: $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t = \mathcal{C}^F \cup \mathcal{C}^B$

Timelike coordinate region:

$$\mathcal{C}_t = \mathcal{C}_t^F \cup \mathcal{C}^B, \quad \Theta(\mathcal{C}_t^F) = \Theta(\mathcal{C}^B)$$

Back coordinate domain:

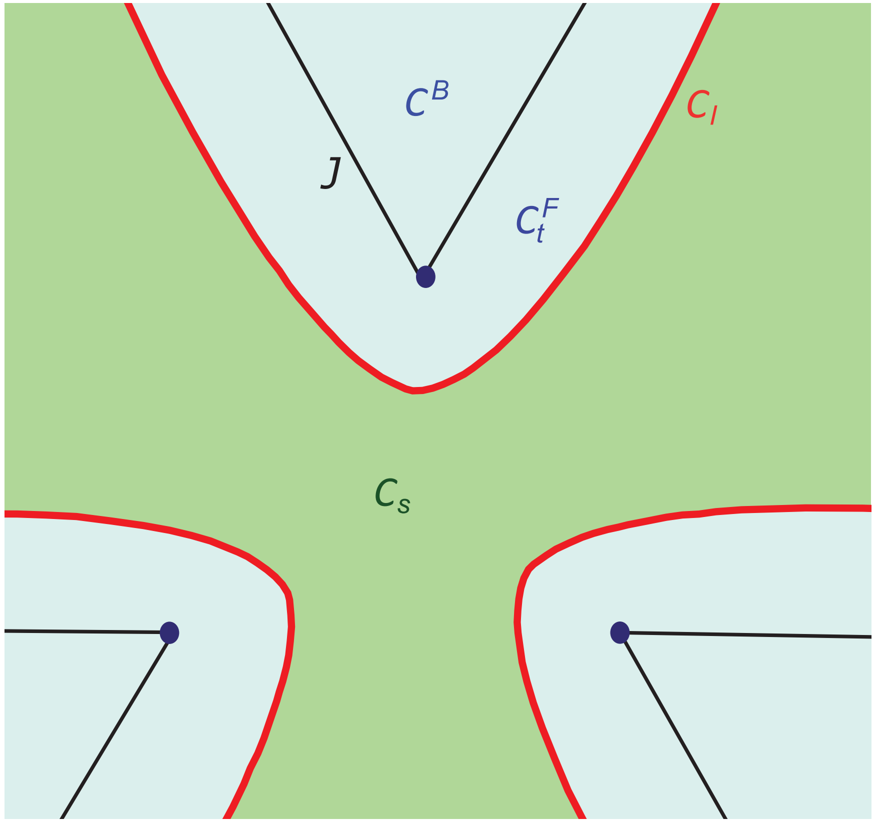
$$\mathcal{C}^B = \mathcal{C}_t - \mathcal{C}_t^F$$

Front coordinate domain:

$$\mathcal{C}^F = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t^F$$

Central region:

$$\mathcal{C}^C = \mathcal{C}_s \cup \mathcal{C}_\ell$$



3D static situation

- The orientation $\hat{\epsilon} \equiv \textit{sgn} * (m_1 \wedge m_2 \wedge m_3 \wedge m_4)$ is the sign of the Jacobian determinant $j_\Theta(x)$.

- $\hat{\epsilon}$ is constant on each emission coordinate domain $\mathcal{C}^F$ and $\mathcal{C}^B$.

- In the central region $\mathcal{C}^C \equiv \mathcal{C}_s \cup \mathcal{C}_\ell$, the orientation $\hat{\epsilon}$ is obtainable from the data $\{\tau^A, \gamma_A(\tau^A)\}$:

$$\forall x \in \mathcal{C}^C, \quad \hat{\epsilon} = \textit{sgn}\,(u \cdot \chi).$$

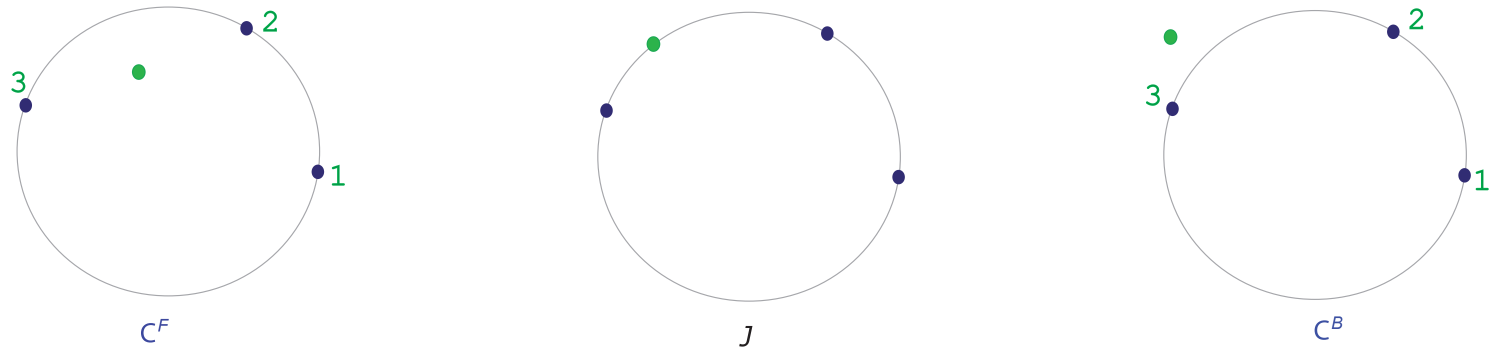
for any future pointing time-like vector u. Then, the transformation is

$$x = \gamma_4 + y_* - \frac{y_*^2\,\chi}{(y_* \cdot \chi) + \textit{sgn}\,(u \cdot \chi)\sqrt{(y_* \cdot \chi)^2 - y_*^2 \chi^2}}$$

- Can the users in the timelike region $\mathcal{C}_t$ know the orientation?

The orientation $\hat{\epsilon}$

- The events where $j_\Theta(x) = 0$ are those for which any user in them can see the four emitters on a circle in his celestial sphere (Coll and Pozo 2005).

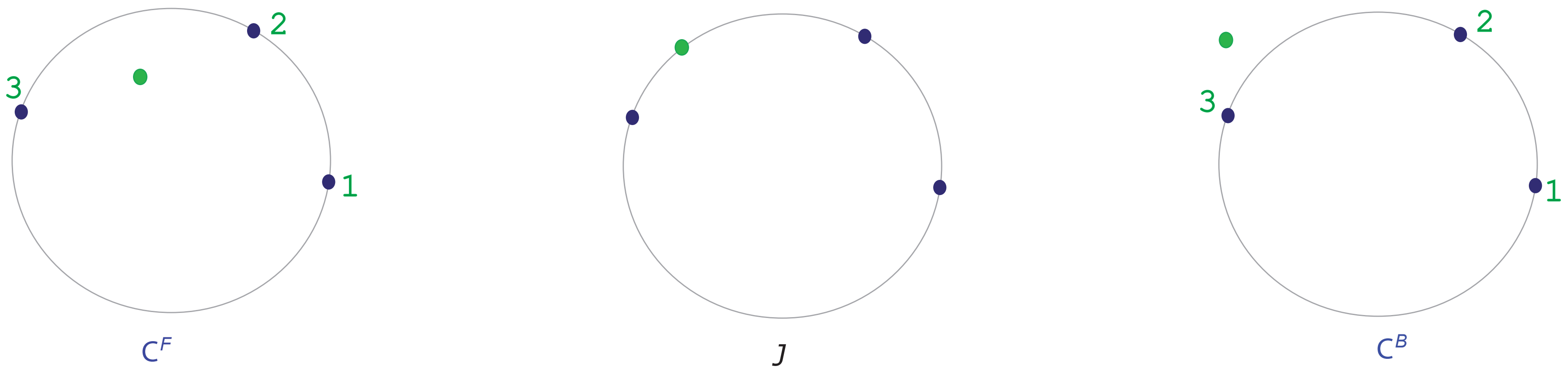


$$\hat{\epsilon} = sgn\Big(*_u \ (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3) \big[i(\overline{m}_4)(\bar{L}^1 + \bar{L}^2 + \bar{L}^3) - 1 \big] \Big)$$

$$m_A = -\epsilon(u \cdot m_A)(u + \overline{m}_A), \quad \bar{L}^a = \frac{\epsilon^{abc} *_u (\overline{m}_b \wedge \overline{m}_c)}{2 *_u (\overline{m}_1 \wedge \overline{m}_2 \wedge \overline{m}_3)}$$

$\{\bar{L}^a\}$ and $\{\overline{m}_a\}$ are dual each other, $\bar{L}^a(\overline{m}_b) = \delta^a_b$.

The orientation $\hat{\epsilon}$



$\hat{\epsilon} = +1$ $\qquad$ $\hat{\epsilon} = -1$

- Observational method to determine $\hat{\epsilon}$:
  - Consider the oriented half cone determined by the lines of sight of three emitters, and then, to look for the position of the other emitter.
  - The orientation is positive (negative) if the line of sight of this fourth emitter is interior (exterior) to the above half cone.

Summary and comments

- We have outlined a method to obtain the orientation of a relativistic positioning system allowing to determine the user's space-time location in inertial frame.

- To localize the users of a GNSS, several geometric methods and algebraic algorithms [Bancroft (1985), Kreuse (1987), Chaffee and Abel (1994), ...] has been developed in the past which are still in use.

- Relativistic positioning concepts has been recently implemented in an algorithm to obtain the Schwarzschild coordinates of a user from his emission coordinates,

  - P. Delva, U. Kostić and A. Čadež *Numerical modeling of a Global Navigation satellite System in a general relativistic framework* Adv. Space Research (2010) arXiv:1005.0477[gr-qc]

- In solving numerically the GNSS navigation equations, one usually pick out an approximate zero order solution.

- In flat space-time, the expression

$$x = \gamma_4 + y_* - \frac{y_*^2\,\chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2\chi^2}}$$

  is the covariant solution of the location problem. For weak gravitational fields it provides the exact non-perturbed zero order solution.

- A numerical analysis of this solution has been accomplished by N. Puchades and D. Sáez (see Puchades's talk that follows).