New examples of marginally trapped surfaces and tubes in warped spacetimes

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Preliminaries

We consider:

- (M^4, g) a 4-dim. spacetime.
- S a compact, without boundary, embedded spacelike surface in (M^4, g).
- $\vec{k}$, $\vec{l}$: two normal, future-pointing lightlike vector fields s.t. $g(\vec{l}, \vec{k}) = -1$.
- $A_{\vec{k}}$ and $A_{\vec{l}}$: associated shape operators.

S is called Marginally Outer Trapped Surface (MOTS)

when $\mathrm{trace}(A_{\vec{l}}) = 0$ and $\mathrm{trace}(A_{\vec{k}}) \neq 0$ everywhere, or viceversa.

Note: MOTS $\Longrightarrow \|\vec{H}\| = 0$, but the converse does not hold.

A Marginally Outer Trapped Tube (MOTT)

is a 3-dimensional smooth manifold $\mathcal{G}$ which admits a foliation by surfaces $\{S_\lambda : \lambda \in \Lambda\}$ s.t. there is a smooth immersion $\Phi : \mathcal{G} \to M^4$ satisfying:

1. each $\Phi(S_\lambda)$ $(\lambda \in \Lambda)$ is a MOTS in M^4,
2. $\Phi(S_\lambda) \cap \Phi(S_\mu) = \emptyset$ for any $\lambda \neq \mu$.

The causal character of the MOTT may vary from point to point.

Targets of the talk

To obtain new examples MOTS and MOTT in a closed Friedman-Lemaître-Robertson-Walker 4-spacetime.

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Firstly, by using CMC surfaces in $\mathbb{S}^3$.

Secondly, by using the classical Hopf map.

CMC surfaces in $\mathbb{S}^3$

Let us consider:

- a smooth function $f : I \subset \mathbb{R} \to (0, \infty)$, $t \in I$,
- a 3-dim. Riemannian manifold (M^3, g_3),
- the Generalized-Robertson-Walker 4-spacetime $\overline{M}_1^4 = I \times M^3$ with line element $\overline{g}_4 = -dt^2 + f^2 g_3$, ($f =$scale factor!)
- a surface S and an immersion,

$$S \xrightarrow{\varphi} M^3$$

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- a surface S and an immersion, and for a fixed $t_o \in I$,

$$\begin{array}{ccccc} S & \xrightarrow{\varphi} & M^3 & \xrightarrow{\psi} & \overline{M}_1^4 \\ & & p & \mapsto & (t_0, p) \end{array} \Rightarrow \phi := \psi \circ \varphi$$

being ϕ an immersion of S in $\overline{M}_1^4$ in the $t = t_o$ slice.

Recall

$$S \xrightarrow{\varphi} M^3 \xrightarrow{\psi} \overline{M}^4_1, \quad \phi := \psi \circ \varphi.$$

If $\vec{H}_\phi$ and $\vec{H}_\varphi$ stand for the mean curvature vectors associated with ϕ and φ, respectively, one obtains

$$\vec{H}_\phi(p) = \frac{\vec{H}_\varphi(p)}{f^2(t_o)} + \frac{f'(t_o)}{f(t_o)}\, \partial_t|_{(t_o,p)}\,. \tag{1}$$

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Theorem

A surface $\phi : S \to (\overline{M}_1^4, -dt^2 + f^2 g_3)$ contained in a t_0-slice satisfies $\|\vec{H}_\phi\| = 0 \iff \varphi : S \to M^3$ has constant mean curvature with $\|\vec{H}_\varphi\| = |f'(t_0)|$.

Corollary

There exist MOTS with arbitrary genus in closed ($M^3 = \mathbb{S}^3$) FLRW spacetimes.

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H.B. Lawson , *Complete Minimal Surfaces in* $\mathbb{S}^3$, Annals of Math. **92** (1970) 335-374.

A. Butscher, F. Pacard, *Doubling Constant Mean Curvature Tori in* $\mathbb{S}^3$, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (5) Vol. V (2006), 611-638.

Proof: By the fact that there exist embedded, compact surfaces with (small) constant mean curvature and arbitrary genus in $\mathbb{S}^3$, we can obtain MOTS with arbitrary genus in closed FLRW $(I \times \mathbb{S}^3, -dt^2 + f^2 g_3)$ (in $t = t_o$ slices).

Examples of MOTT in closed FLRW foliated by tori with different causality

Let $\mathbb{C}$ be the complex numbers, with $i = \sqrt{-1}$, $|z|$ the modulus of $z \in \mathbb{C}$, $\bar{z}$ its complex conjugate.

$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, with standard metric g_3.

Recall the CMC embedded torus C_u in $\mathbb{S}^3$ given by

$$C_u := \left\{(z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2 : |z_1| = \cos(u),\ |z_2| = \sin(u)\right\},$$

$u \in (0, \pi/2)$, with mean curvature $\|\vec{H}_u\| := |2\cot(2u)|$.

Define

$$h : I \to (0, \pi/2), \ \ h(t) = \frac{1}{2}\operatorname{arccot}\left(\frac{f'(t)}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{f'(t)}{2}\right).$$

And now, the embedding

$$\chi : I \times \mathbb{S}^1 \times \mathbb{S}^1 \to -I \times_f \mathbb{S}^3,$$
$$\chi(t, e^{i\theta}, e^{i\nu}) = \left(t, e^{i\theta}\cos(h(t)), e^{i\nu}\sin(h(t))\right).$$

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For each $t \in I$, the surface $\phi = \chi(t, -, -) : \mathbb{S}^1 \times \mathbb{S}^1 \to \overline{M}^4_1$, is a torus, embedded in the t-slice, with constant mean curvature $\|\vec{H}_\phi\| = \left|2\cot(2u)\,|_{u=h(t)}\right| = |f'(t)|$.

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For each $t \in I$, the surface $\phi = \chi(t, -, -) : \mathbb{S}^1 \times \mathbb{S}^1 \to \overline{M}_1^4$, is a torus, embedded in the t-slice, with constant mean curvature $\|\vec{H}_\phi\| = \left|2\cot(2u)\,|_{u=h(t)}\right| = |f'(t)|$.

By our theorem, each torus is a MOTS, and therefore, χ is a MOTT.

The induced metric is:

$$\chi^*\bar{g}_4 \equiv \begin{pmatrix} \bar{g}_4(\chi_t, \chi_t) & 0 & 0 \\ 0 & (f(t)\cos(h(t)))^2 & 0 \\ 0 & 0 & (f(t)\sin(h(t)))^2 \end{pmatrix},$$

The causal character depends only on χ_t:

$$z(t) := \bar{g}_4(\chi_t, \chi_t) = -1 + \left(\frac{f(t)f''(t)}{4 + f'(t)^2}\right)^2 .$$

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1. Given $a, b > 0$ such that $a^2 = 4 + b^2$, define the function $f : I = \mathbb{R} \to (0, \infty)$, $f(t) = a \cosh(t) + b \sinh(t)$. Then, $z(t) \equiv 0$. Therefore, χ_t is everywhere lightlike.

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2. Define the function $f : (-1, 1) \to (0, \infty)$, $f(t) = \dfrac{2}{1 - t^2}$. By simple computations, we obtain $z(t) \geqslant 3$, for any $t \in (-1, 1)$, and therefore χ_t is always spacelike.

3. Take real constants $c_1, c_2 > 0$. Then, the function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \frac{4+c_1^2}{4c_2} t^2 + c_1 t + c_2$ is well-defined. A simple computation shows $z(t) \equiv -3/4$. This implies that χ_t is everywhere timelike.

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4. Given the function $f : \mathbb{R} \to (0, \infty)$, $f(t) = 3 + \cos(2t)$. A straightforward computation gives

$$z(t) := -1 + \left(\frac{f(t) f''(t)}{4 + f'(t)^2} \right)^2 = -1 + \frac{4\cos^2(2t)(3+\cos(2t))^2}{(3-\cos(4t))^2}.$$

Finally, it is easy to check $z(0) = 15$ and $z(\pi/4) = -1$. In this case, the causal character changes *with time*.

The classical Hopf map

$\mathbb{S}^2(1/2) = \{(z,x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1/4\}$, with standard metric g_2 (of radius $1/2$.

The classical **Hopf map** is

$$\pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2), \quad \pi(z,w) = \left(z\overline{w}, \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 \right).$$

1. π is a Riemannian submersion.
2. For each $(z,a) \in \mathbb{S}^2(1/2)$, then $\pi^{-1}\{(z,a)\} =$ closed geodesic in $\mathbb{S}^3$.

Extending π to a new submersion

$$\begin{array}{c}(\mathbb{S}^3, g_3)\\ \Big\downarrow \pi \\ (\mathbb{S}^2(1/2), g_2)\end{array}$$

Extending π to a new submersion

$$\begin{array}{ccc} (\mathbb{S}^3, g_3) & (-I \times_f \mathbb{S}^3, -dt^2 + f^2 g_3) & (t, p) \\ \Big\downarrow \pi & \Big\downarrow \overline{\pi} = Id \times \pi & \Big\downarrow \\ (\mathbb{S}^2(1/2), g_2) & (-I \times_f \mathbb{S}^2(1/2), -dt^2 + f^2 g_2) & (t, \pi(p)) \end{array}$$

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Next, we consider a curve α in $-I \times_f \mathbb{S}^2(1/2)$,

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\begin{array}{ccc}
 & & -I \times_f \mathbb{S}^3 \\
 & & \Big\downarrow \overline{\pi} \\
J \subset \mathbb{R} & \xrightarrow{\alpha} & -I \times_f \mathbb{S}^2(1/2)
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Next, we consider a curve α in $-I \times_f \mathbb{S}^2(1/2)$, **and its pullback**:

$$
\begin{array}{ccc}
\overline{\pi}^* \alpha = J \times \mathbb{S}^1 & \longrightarrow & -I \times_f \mathbb{S}^3 \\
\Big\downarrow & & \Big\downarrow \overline{\pi} \\
J \subset \mathbb{R} & \xrightarrow{\alpha} & -I \times_f \mathbb{S}^2(1/2)
\end{array}
$$

The geometric elements of α determine the properties of the mean curvature vector of $\pi^* \alpha$.

$$\begin{array}{ccc} \overline{\pi}^*\alpha = J \times \mathbb{S}^1 & \longrightarrow & -I \times_f \mathbb{S}^3 \\ \downarrow & & \downarrow \overline{\pi} \\ J \subset \mathbb{R} & \xrightarrow{\alpha} & -I \times_f \mathbb{S}^2(1/2) \end{array}$$

For instance, if α is embedded and open / closed, then $\overline{\pi}^*(\alpha)$ is an embedded cylinder / torus in $-I \times_f \mathbb{S}^3$.

These surfaces may not be contained in a single t-slice.

Consider a unit spacelike Frenet curve.

$$\begin{array}{ccc} & & \left(-I \times_f \mathbb{S}^3, -dt^2 + f^2 g_3\right) \\ & & \Big\downarrow \overline{\pi} \\ J & \xrightarrow{\alpha} & \left(-I \times_f \mathbb{S}^2(1/2), -dt^2 + f^2 g_2\right) \end{array}$$

Consider a unit spacelike Frenet curve. Let β be a **horizontal lift** of α.

$$\begin{array}{ccc} J & \xrightarrow{\beta} & \left(-I \times_f \mathbb{S}^3, -dt^2 + f^2 g_3\right) \\ \| & & \downarrow \overline{\pi} \\ J & \xrightarrow{\alpha} & \left(-I \times_f \mathbb{S}^2(1/2), -dt^2 + f^2 g_2\right) \end{array}$$

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Also, for each $e^{i\theta} \in \mathbb{S}^1$, the map

$$\overline{\Gamma}_\theta : -I \times_f \mathbb{S}^3 \to -I \times_f \mathbb{S}^3, \quad \overline{\Gamma}_\theta(t,(z,w)) = \left(t,(e^{i\theta}z, e^{i\theta}w)\right)$$

is an isometry. Now, define:

$$\phi : \overline{\pi}^*(\alpha) = J \times \mathbb{S}^1 \to -I \times_f \mathbb{S}^3, \quad \phi(s,\theta) = \overline{\Gamma}_\theta(\beta(s)).$$

ϕ is just a parametrization of the surface $\overline{\pi}^*(\alpha)$.

If $\overline{g}_3 = -dt^2 + f^2 g_2$ is the line element, let $\alpha : J \subset \mathbb{R} \to -I \times_f \mathbb{S}^2(1/2)$ be a unit spacelike Frenet curve with Frenet apparatus $\{T = \dot{\alpha}, N, B\}$ and κ, τ, i.e.

$$\nabla_T T = \epsilon_2 \kappa N, \quad \nabla_T N = \kappa T + \epsilon_3 \tau B, \quad \nabla_T B = -\epsilon_2 \tau N,$$

where $\epsilon_2 = \overline{g}_3(N, N)$, $\epsilon_3 = \overline{g}_3(B, B)$, $\epsilon_2 = -\epsilon_3 = \pm 1$, and $\{T, N, B\}$ is a positive basis along α.

$$\begin{array}{ccc} \overline{\pi}^*(\alpha) = J \times \mathbb{S}^1 & \xrightarrow{\ \phi\ } & (-I \times_f \mathbb{S}^3, \overline{g}_4 = -dt^2 + f^2 g_3) \\ \downarrow & \overset{\beta}{\nearrow} & \downarrow \overline{\pi} \\ J & \xrightarrow{\ \alpha\ } & (-I \times_f \mathbb{S}^2(1/2), \overline{g}_3 = -dt^2 + f^2 g_2) \end{array}$$

Let $\tilde{N}$ and $\tilde{B}$ be horizontal lifts of N and B, resp., along β.

Lemma

The mean curvature vector of ϕ is given by

$$\vec{H}_\phi = \frac{\epsilon_2}{2}\left(\kappa + \frac{f'}{f}\overline{g}_3(\partial_t, N)\right)(\overline{\Gamma}_\theta)_*\tilde{N} + \frac{\epsilon_3}{2}\left(\frac{f'}{f}\overline{g}_3(\partial_t, B)\right)(\overline{\Gamma}_\theta)_*\tilde{B}$$

Proposition

The mean curvature vector $\vec{H}_\phi$ satisfies $\|\vec{H}_\phi\| = 0$ iff

$$\left(\kappa + \frac{f'}{f}\overline{g}_3(\partial_t, N)\right)^2 - \left(\frac{f'}{f}\overline{g}_3(\partial_t, B)\right)^2 = 0.$$

Final remarks

We obtained an **open** embedded surface with null mean curvature vector, and crossing two regions (expanding and conllapsing).

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Open problem

To obtain an explicit MOTS in the 4-dim closed FLRW spacetime, which is not contained in any t-slice, from a closed curve in the toy model $-\mathrm{I} \times_f \mathbb{S}^2(1/2)$.

Conclusions

- There exist MOTS in closed FLRW 4-spacetimes embedded in t_o-slices with arbitrary topology.
- This leads to MOTT in closed FLRW 4-spacetimes.
- From a curve in a (toy model) closed FLRW 3-spacetime $\alpha : J \to (-I \times_f \mathbb{S}^2(1/2), -dt^2 + f^2 g_2)$, it is possible to construct embedded cylinders and tori in the closed FLRW 4-spacetime $(-I \times_f \mathbb{S}^3, -dt^2 + f^2 g_3)$ with some control of the mean curvature vector.
- Problem: to construct such a tori which is also a MOTS.

J. L. Flores, S. Haesen, M. Ortega, *Class. Quantum Grav.* **27** (2010)

Thank you very much
for your attention!!