Black-hole lasers
in Bose–Einstein condensates

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ERE 2010 Granada

PRD 81, 0840242 (2010) AC + RP,
arXiv:1005.4024 *to appear in NJP* SF + RP,
PRA 80, 043601 (2009) J.Macher + RP.

Plan

- I. Black hole instabilities: a brief review.
- II. Phonon mode equation in BEC: exact eq.,
  no hydrodynamical approxim.
- III. Black holes in BEC.
  - **Phonon spectra** in **supersonic** flows with
    - **one** sonic BH **or** WH horizon,
    - **a pair** of BH **and** WH horizons.
  - Impact of the **second** horizon on **observables**.
  - **Classical** vs **Quantum** description of **dyn. instabilities**.

Black hole instabilities, 1.

- The **stability** of the Schwarzschild Black Hole

$$ds^2 = -(1 - \frac{r_S}{r})\,dt^2 + \frac{dr^2}{(1 - \frac{r_S}{r})} + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

with $r_S = 2GM/c^2$, was a subject of **controversy** $\rightarrow$ 50's.

- **stability** demonstrated by Wheeler (and others), i.e., The **spectrum** of metric perturbations contains **no complex frequency asympt. bound modes**

- its (astro)-physical relevance recognized.

Black hole instabilities, 2.

A **rotating Black Hole** (Kerr) is subject to a **weak instability**:

- **Classical** waves display a **super-radiance**:

$$\phi^{\text{in}}_{\omega,l,m} \to R_{\omega,l,m}\, \phi^{\text{out}}_{\omega,l,m} + T_{\omega,l,m}\, \phi^{\text{absorbed}}_{\omega,l,m},$$

  with

$$|R_{\omega,l,m}|^2 > 1.$$

  Energy is **extracted** from the hole.
  This is a **stimulated** process.

- At the **Quantum** level, **super-radiance implies** a **steady spontaneous** pair creation process, i.e. a **"vacuum instability"**.

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Black hole instabilities, 3.

- When introducing a **reflecting boundary condition**, the **super-radiant instability** induces a **dynamical instability** a **Black Hole Bomb**, Press ’70, Kang ’97, Cardoso et al ’04.
- A **non-zero mass** can induce a **reflection**, Damour et al ’76; this is presently used to constrain the mass of ’axions’.
- As in a resonant cavity, the **spectrum** now contains a **discrete set** of modes with **complex** frequencies.

Black hole instabilities, 4.
The superluminal Black Hole Laser

- discovered by Corley & Jacobson in 1999,
- arises in the presence of **two** horizons (charged BH) **and** with **superluminal** **dispersion**,
- the 'trapped' region acts as a **cavity**,
- induces an **exponential growth** of **Hawking radiation**, and constitutes a **dynamical instability**.

- **Naturally** arises in **Acoustic Black Holes**: **supersonic flows** in Bose Einstein condensates,
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Black hole lasers (in BEC)

- discovered by Corley & Jacobson in 1999,
- "revisited" by Leonhardt & Philbin in 2008,
- twice studied in terms of **time-dep. wave-packets**,
- instead, in what follows,
  a **spectral analysis** of **statio. modes**.
- see also
  Garay et al. PRL 85 and PRA 63 (2000/1), BH/WH flows in BEC
  Barcelo et al. PRD 74 (2006), Dynam. stab. analysis
  and Jain et al. PRA 76 (2007). Quantum De Laval nozzle

Bose Einstein Condensates

- Set of atoms is described by $\hat{\Psi}(t, \mathbf{x})$ obeying

$$[\hat{\Psi}(t, \mathbf{x}), \hat{\Psi}^\dagger(t, \mathbf{x}')] = \delta^3(\mathbf{x} - \mathbf{x}'),$$

and by a Hamiltonian

$$\hat{H} = \int d^3x \left\{ \frac{\hbar^2}{2m} \nabla_{\mathbf{x}} \hat{\Psi}^\dagger \nabla_{\mathbf{x}} \hat{\Psi} + V(x) \hat{\Psi}^\dagger \hat{\Psi} + \frac{g(x)}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right\}.$$

- at low temperature, $\hat{\Psi}$ is expanded as

$$\begin{aligned} \hat{\Psi}(t, \mathbf{x}) &= \Psi_0(t, \mathbf{x}) + \hat{\psi}(t, \mathbf{x}) \\ &= \Psi_0(t, \mathbf{x}) (1 + \hat{\phi}(t, \mathbf{x})), \end{aligned} \quad (1)$$

$\Psi_0(t, \mathbf{x})$ describes the **condensed atoms**,
$\hat{\psi}$ ($\hat{\phi}$) describes **(relative) perturbations**.

1D statio condensates

A 1D stationary condensate is described by

$$\Psi_0(t,x) = e^{-i\mu t/\hbar} \times \sqrt{\rho_0(x)}\, e^{i\theta_0(x)},$$

ρ_0 is the **mean density** and $v = \frac{\hbar}{m}\partial_x \theta_0$ the **mean velocity**.

ρ_0, v are determined by V and g through
the **Gross Pitaevskii** eq.

$$\mu = \frac{1}{2} m v^2 - \frac{\hbar^2}{2m} \frac{\partial_x^2 \sqrt{\rho_0}}{\rho_0} + V(x) + g(x)\,\rho_0, \qquad (2)$$

which also gives

$$\partial_x (v \rho_0) = 0. \qquad (3)$$

BdG equation for relative density fluctuations

- In a BEC, density fluctuations obey the **BdG equation**. For relative fluctuations, this eq. is

$$i\hbar(\partial_t + v\partial_x)\hat{\phi} = \left[T_v + mc^2\right]\hat{\phi} + mc^2\hat{\phi}^\dagger, \quad (4)$$

$$c^2(x) \equiv \frac{g(x)\rho_0(x)}{m}, \quad (5)$$

is the x-dep. **speed of sound** and T_v a kinetic term

$$T_v \equiv -\frac{\hbar^2}{2m} v\partial_x \frac{1}{v}\partial_x. \quad (6)$$

- **Only** v and c enter in **BdG eq.**: V, g and $\frac{\partial_x^2\sqrt{\rho_0}}{\rho_0}$ drop out. **Exact result**, no **hydro.**, no eikonal approximation.

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Covariantizing the BdG equation ?

Since phonons

- only see the macrosc. **mean** fields $c(x), \mathbf{v}(x), \rho_0(x)$,
- are **insensitive** to microsc. qtts $g(x)$, $V(x)$ and Q.pot.

this allows:

- to forget about the (fundamental) theory of the condensate, when **computing the phonon spectrum**.
- to consider the phonon field from a **4D point of view** by **covariantizing** the BdG eq. introducing **4D tensors**
  - the (Unruh) acoustic metric $g_{\mu\nu}(t,x)$
  - the (Jacobson) unit time-like vector field $u^\mu(t,x)$
  - extra scalars ...

Not just an **analogy**, but an **equivalent point of view**.

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Computing phonon spectra. 1.

- basically **equivalent** to that of a **hermitian scalar field**.
- to handle the complex character of $\hat{\phi}$, it is useful (Leonhardt et al. '03) to introduce the **doublet**

$$\hat{W} \equiv \begin{pmatrix} \hat{\phi} \\ \hat{\phi}^\dagger \end{pmatrix}, \tag{7}$$

invariant under a **pseudo-Hermitian conjugation** ($pH.c.$)

$$\hat{W} = \bar{\hat{W}} \equiv \sigma_1 \hat{W}^\dagger. \tag{8}$$

- The mode decomposition of $\hat{W}$ is

$$\hat{W} = \sum_n (W_n\, \hat{a}_n + \bar{W}_n\, \hat{a}_n^\dagger) = \sum_n (W_n\, \hat{a}_n + pH.c.), \tag{9}$$

where the modes $W_n(t,x)$ are doublets of $\mathbb{C}$-functions.

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Computing spectra. 2. The inner product

- The **conserved inner product**

$$\langle W_1 | W_2 \rangle \equiv \int \mathrm{d}x \, \rho_0(x) \, W_1^*(t,x) \, \sigma_3 \, W_2(t,x), \qquad (10)$$

is **not** **positive definite** (c.f. the Klein-Gordon product).

- **As usual**, mode orthogonality

$$\langle W_n | W_m \rangle = -\langle \bar{W}_n | \bar{W}_m \rangle = \delta_{nm}, \qquad (11)$$

implies canonical commutators

$$[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}, \qquad (12)$$

where

$$\hat{a}_n = \langle W_n | \hat{W} \rangle. \qquad (13)$$

Computing spectra.
3. The notion of Asympt. Bound Modes

For stationary backgrounds with **infinite spatial extension** the **solutions** of

$$H\, W_\lambda(x) = \lambda\, W_\lambda(x), \tag{14}$$

which **belong to the spectrum must be Asymptotically Bound**: bound for $x \to \pm\infty$.

- N.B.1. Hence, in certain **non-homogeneous** backgrounds, **the freq.** λ can be **complex**.
- N.B.2. **Quasi Normal Modes** are **not ABM**, hence are **not** in the spectrum.

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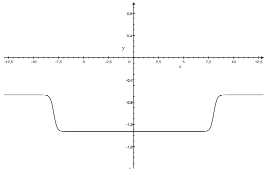
The background stationary profiles

Flows with **one** or **two** **sonic horizons**, $c = |v|$:
That is, $v(x) < 0$ and, for **one** horizon:

$$c(x) + v(x) = c_{\mathrm{H}} D \tanh\left(\frac{\kappa_{\mathrm{B}} x}{c_{\mathrm{H}} D}\right),$$

where $\kappa_{\mathrm{B}} = \partial_x (c + v)|_{\mathrm{hor.}}$, Carter's **decay rate** $\sim$ surf. gravity,
and for **two** horizons:

$$c(x) + v(x) = c_{\mathrm{H}} D \tanh\left(\frac{\kappa_{\mathrm{W}}(x + L)}{c_{\mathrm{H}} D}\right) \tanh\left(\frac{\kappa_{\mathrm{B}}(x - L)}{c_{\mathrm{H}} D}\right),$$



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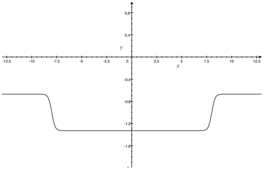

Spectrum of W_n for one B/W sonic horizon

The **complete set of modes** is (Macher-RP 2009)

- a **continuous** set of **real frequency** modes which contains
- for $\omega > \omega_{\rm max}$, **two positive** norm modes, as in flat space, W^u_ω, W^v_ω, which resp. describe right/left moving phonons,
- for $0 < \omega < \omega_{\rm max}$, **three** modes: 2 positive norm W^u_ω, W^v_ω + 1 **negative** norm mode $\bar{W}^u_{-\omega}$.
- The **threashold** freq. $\omega_{\rm max}$ scales $1/\text{healing length} = mc/\hbar$, but **also** depends on $D = (v_{\rm asympt.} + c_{\rm asympt.})/c_H$.

- **Lessons:**
  - There are **no complex** freq. **ABM**,
  - **Same spectrum** for **White Holes** and **Black Holes**, because **invariant** under $v \to -v$.
  - Hence **White Hole** flows are **dyn. stable**, as **BH** ones.

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The scattering of *in*-modes

- For $\omega > \omega_{\max}$, there is an **elastic** scattering:

$$W_\omega^{u,\,in} = T_\omega W_\omega^{u,\,out} + R_\omega W_\omega^{v,\,out}, \quad \text{with,}$$
$$|T_\omega|^2 + |R_\omega|^2 = 1. \tag{15}$$

- For $0 < \omega < \omega_{\max}$, there is a $3{\times}3$ matrix, e.g.

$$W_\omega^{u,\,in} = \alpha_\omega W_\omega^{u,\,out} + R_\omega W_\omega^{v,\,out} + \beta_\omega \bar{W}_{-\omega}^{u,\,out}, \tag{16}$$

with

$$|\alpha_\omega|^2 + |R_\omega|^2 - |\beta_\omega|^2 = 1. \tag{17}$$

- The β coefficients describe a **super-radiance**, hence a **vacuum instability** in QM, i.e. the **spontaneous** sonic B/W hole radiation.

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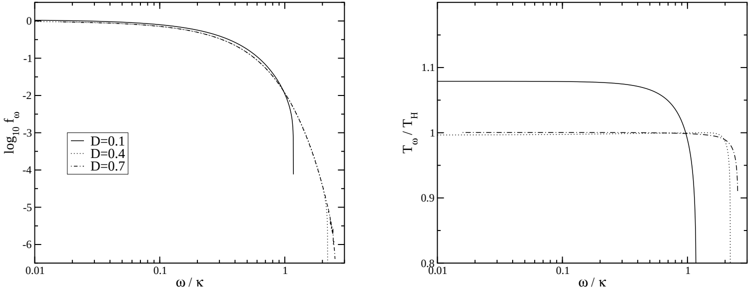
The (numerical) properties of this radiation

For $\omega_{\rm max} \geq 3\kappa$, the energy spectrum $f_\omega = \omega \, |\beta_\omega|^2$ is (JM-RP '09)

- Planckian (up to $\omega_{\rm max}$) and
- with a temperature $= \kappa/2\pi = T_{\rm Hawking}$, $\quad (f_\omega = \omega/(e^{\omega/T_\omega} - 1)$,

"exactly" as **predicted** by the **gravitational analogy**.



N.B. The above spectra are obtained from the BdG eq. **only**.

Spectrum of W_n for two sonic horizons

The (complete) set of modes contains (AC+RP 2010)

- a **continuous** spectrum of **real** freq. modes W_ω^u, W_ω^v with $0 < \omega < \infty$, with **positive norm only**, and of dim. **2**.
- a **discrete** set of **pairs** of **complex** freq. modes (V_a, Z_a) with cc freq. (λ_a, λ_a^*), where $a = 1, 2, ...N < \infty$.

N.B. Negative norm modes $\bar{W}_{-\omega}$ are **no longer** in the spectrum; hence there is **no** Bogoliubov transformation in the present case.

The field operator thus reads

$$\hat{W} = \int_0^\infty d\omega \sum_{\alpha=u,v} \left[e^{-i\omega t} W_\omega^\alpha(x)\, \hat{a}_\omega^\alpha + pH.c. \right] + \sum_a \left[e^{-i\lambda_a t} V_a(x)\, \hat{b}_a + e^{-i\lambda_a^* t} Z_a(x)\, \hat{c}_a + pH.c. \right]. \quad (18)$$

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Norms and commutators

- The **real** freq., the modes W_ω^α and operators $\hat{a}_\omega^\alpha$ obey

$$\langle W_\omega^\alpha | W_{\omega'}^{\alpha'} \rangle = \delta(\omega - \omega')\delta_{\alpha\alpha'} = -\langle \bar{W}_\omega^\alpha | \bar{W}_{\omega'}^{\alpha'} \rangle \tag{19}$$

and

$$[\hat{a}_\omega^\alpha, \hat{a}_{\omega'}^{\alpha' \dagger}] = \delta(\omega - \omega')\delta_{\alpha\alpha'}. \tag{20}$$

- Instead for **complex frequency** λ_a, one has

$$\langle V_a | V_{a'} \rangle = 0 = \langle Z_a | Z_{a'} \rangle, \quad \langle V_a | Z_{a'} \rangle = i\delta_{aa'}, \tag{21}$$

and

$$[\hat{b}_a, \hat{b}_{a'}^\dagger] = 0, \quad [\hat{b}_a, \hat{c}_{a'}^\dagger] = i\delta_{aa'}. \tag{22}$$

The two-mode sectors with complex freq. λ_a

Each pair $(\hat{b}_a, \hat{c}_a)$ **always** describes
one complex, rotating, unstable oscillator:

- Its (Hermitian) Hamiltonian is

$$\hat{H}_a = -i\lambda_a \hat{c}_a^\dagger \hat{b}_a + H.c. \tag{23}$$

- Writing

$$\lambda_a = \omega_a + i\Gamma_a, \tag{24}$$

with ω_a, Γ_a real > 0,
$\Re\lambda_a = \omega_a$ fixes the angular velocity,
$\Im\lambda_a = \Gamma_a$ fixes the **growth rate**.

Computing the spectrum of ABM

The method:

- **A.** use WKB waves to
  - 1. **decompose** the exact modes,
  - 2. obtain **algebraic relations** (valid **beyond WKB**) between the $\mathbb{R}$ freq. W_ω and the $\mathbb{C}$ freq. V_a, Z_a
- **B.** a numerical analysis to validate the predictions.

N.B. The W_ω are **deeply connected** to the V_a, Z_a because

$$H\, W_\lambda = \lambda\, W_\lambda \quad (25)$$

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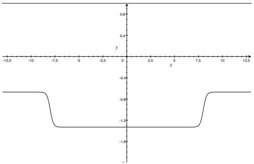
The scattering of real freq. u-mode



- On the **left** of the White hor. $W_\omega^{u,\,in} \to W_\omega^u$, the WKB sol.
- **Between** the two horizons, **for** $\omega < \omega_{\rm max}$,

$$W_\omega^{u,\,in} = \mathcal{A}_\omega\, W_\omega^u + \mathcal{B}_\omega^{(1)}\, \bar{W}_{-\omega}^{(1)} + \mathcal{B}_\omega^{(2)}\, \bar{W}_{-\omega}^{(2)}, \qquad (26)$$

- On the **right** of the Black horizon, $W_\omega^{u,\,in} \to e^{i\theta_\omega}\, W_\omega^u$.
- N.B.1. **Negative norm/freq** WKB modes $\bar{W}_{-\omega}^{(i)}$ in (26). Hence "*anomalous* scattering" ($\sim$ Bogoliubov transf.).
- N.B.2. Modes **fully described** by $\mathcal{A}_\omega, \mathcal{B}_\omega^{(1)}, \mathcal{B}_\omega^{(2)}$ and θ_ω.

Computing $\mathcal{A}_\omega, \mathcal{B}_\omega^{(1)}, \mathcal{B}_\omega^{(2)}$ and θ_ω

- algebraically achieved by introd. a 2-vector $(W_\omega^u, \bar{W}_{-\omega})$, on which acts a 2×2 S**-matrix** (Leonhardt 2008)
- this S-matrix can be decomposed as

$$S = U_4\, U_3\, U_2\, U_1. \qquad (27)$$

where

- U_1 describes the **scattering** on the **WH** horizon.
- U_2 the **propagation from** the WH to the BH
- U_3 the **scattering** on the **BH** horizon.
- U_4 the **escape** to the right of W_ω^u and the **return** of $\bar{W}_{-\omega}^{(2)}$ to the WH horizon.

Computing $\mathcal{A}_\omega, \mathcal{B}_\omega^{(1)}, \mathcal{B}_\omega^{(2)}$ and θ_ω

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$$S = U_4 U_3 U_2 U_1. \tag{27}$$

where

- U_1 describes the **scattering** on the **WH** horizon.
- U_2 the **propagation from** the WH to the BH
- U_3 the **scattering** on the **BH** horizon.
- U_4 the **escape** to the right of W_ω^u and the **return** of $\bar{W}_{-\omega}^{(2)}$ to the WH horizon.

Explicitly,

$$U_1 = S_{WH} = \begin{pmatrix} \alpha_\omega & \alpha_\omega z_\omega \\ \tilde{\alpha}_\omega z^*_\omega & \tilde{\alpha}_\omega \end{pmatrix}, \quad U_2 = \begin{pmatrix} e^{iS^u_\omega} & 0 \\ 0 & e^{-iS^{(1)}_{-\omega}} \end{pmatrix},$$

$$U_3 = S_{BH} = \begin{pmatrix} \gamma_\omega & \gamma_\omega w_\omega \\ \tilde{\gamma}_\omega w^*_\omega & \tilde{\gamma}_\omega \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 \\ 0 & e^{iS^{(2)}_{-\omega}} \end{pmatrix} \tag{28}$$

where

$$S^u_\omega \equiv \int_{-L}^{L} \mathrm{d}x\, k^u_\omega(x), \quad S^{(i)}_{-\omega} \equiv \int_{-L_\omega}^{R_\omega} \mathrm{d}x \left[-k^{(i)}_\omega(x) \right], \quad i = 1, 2, \tag{29}$$

are H-Jacobi actions, and L_ω and R_ω are the two turning points.
By unitarity, one has $|\alpha_\omega|^2 = |\tilde{\alpha}_\omega|^2$, $|\alpha_\omega|^2 = 1/(1 - |z_\omega|^2)$.

The single-valued real freq. mode

The mode $W_\omega^{u,in}(x)$ **must be single-valued.**
Hence the **trapped** piece $\mathcal{B}_\omega^{(2)}$ of
$W_\omega^{u,in} = \mathcal{A}_\omega W_\omega^u + \mathcal{B}_\omega^{(1)} \bar{W}_{-\omega}^{(1)} + \mathcal{B}_\omega^{(2)} \bar{W}_{-\omega}^{(2)}$ must obey

$$\begin{pmatrix} e^{i\theta_\omega} \\ \mathcal{B}_\omega^{(2)} \end{pmatrix} = S \begin{pmatrix} 1 \\ \mathcal{B}_\omega^{(2)} \end{pmatrix}, \qquad (30)$$

which implies

$$\mathcal{B}_\omega^{(2)} = \frac{S_{21}(\omega)}{1 - S_{22}(\omega)}. \qquad (31)$$

The first key equation. (Valid beyond WKB.)

When $Im\,\lambda = \Gamma > 0$, $\rightarrow Im\,k_\lambda^u > 0$, hence **growth** for $x \rightarrow -\infty$.
So any single-valued **ABMode** must satisfy

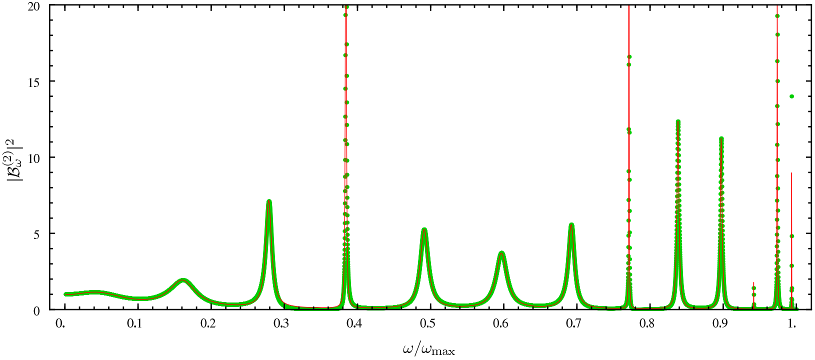
$$\begin{pmatrix} \beta_a(\lambda) \\ 1 \end{pmatrix} = S(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{32}$$

This implies

$$S_{22}(\lambda) = 1, \quad \beta_a = S_{12}(\lambda). \tag{33}$$

Second key result:
The **poles** of $\mathcal{B}_\omega^{(2)}$ **correspond** to the **complex freq.** λ_a.

$|\mathcal{B}_\omega^{(2)}|^2$, as a function of ω **real**.



- Green dots are **numerical values**,
  the continuous red line is a sum of Lorentzians.
- Near a complex frequency λ_a, solution of $S_{22} = 1$,
  $|\mathcal{B}_\omega^{(2)}|^2 \sim C_a / |\omega - \omega_a - i\Gamma_a|^2$, i.e. a Lorentzian.
- Above $\omega_{\max}$ no peaks, because no neg. norm WKB mode.

Computing the complex freq. $\lambda_a = \omega_a + i\Gamma_a$.

- The λ_a's, are fixed by the cond. **ABM + single-valued**. **Both conditions encoded in $S_{22} = 1$**.
- When the **leaking-out amplitudes** are small, $|z_\omega|, |w_\omega| = |\beta_\omega/\alpha_\omega| \ll 1$, the supersonic region acts as a **cavity**:
- To **zeroth order** in z_ω, w_ω, $S_{22} = 1$ fixes $\Re\lambda_a = \omega_a$ by a **Bohr-Sommerfeld** condition

$$S^{(1)}_{-\omega} - S^{(2)}_{-\omega} + \pi = \int_{-L}^{L} dx[-k^{(1)}_\omega(x) + k^{(2)}_\omega(x)] + \pi = 2\pi n,$$

where $n = 1, 2, ..., N$.
This explains the **discreteness** of the set.

To **second order** in z_ω, w_ω, $S_{22} = 1$ fixes $Im\, \lambda_a = \Gamma_a$ to be

$$2\Gamma_a T^b_{\omega_a} = |S_{12}(\omega_a)|^2 = |z_{\omega_a} + w_{\omega_a} e^{i\psi_a}|^2 \tag{34}$$

- $T^b_{\omega_a} > 0$ is the **bounce time**, given by

$$T^b_\omega = \frac{\partial}{\partial \omega} \left(S^{(2)}_{-\omega} - S^{(1)}_{-\omega} + \text{"non HJ terms"} \right) \tag{35}$$

- The phase in the cosine is

$$\psi_a = S^u_{\omega_a} + S^{(1)}_{-\omega_a} + \text{other "non HJ terms"} \tag{36}$$

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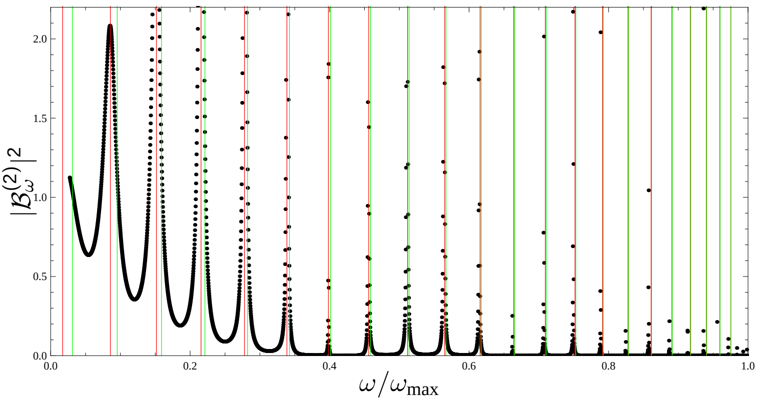
The validity of the 'semi-classical' treatment.

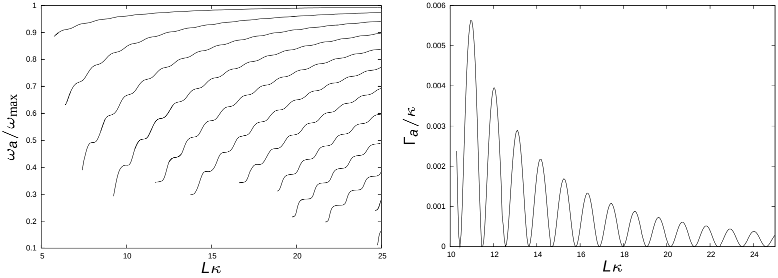


Dots are **numerical values**.

The 22 red lines are the **predictions**.

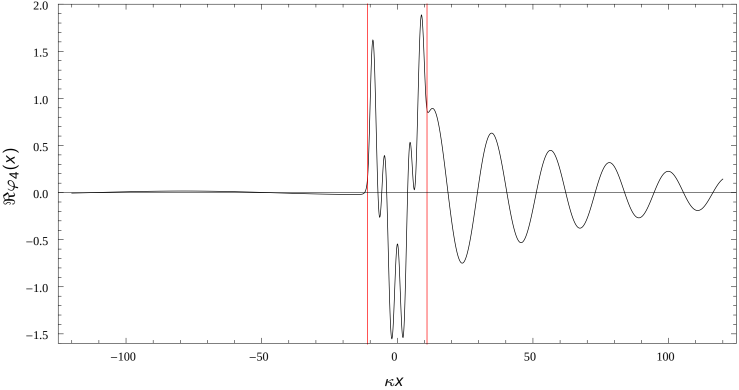
Excellent agreement

The evolution of ω_a and Γ_a in terms of L.



- **New** bound modes appear as L grows, **with** $\omega = \Gamma = 0$ **?**
- The Γ_a reach their maximal value for $\omega_a/\omega_{\max} \ll 1$.
- Γ_a reach 0 because of (Young) interferences.
  The destruction is imperfect when $z_\omega \neq w_\omega$.
- No bound mode is destroyed as L grows.

A typical growing mode with a high Γ_a ($\Gamma/\omega \sim 1/20$)



- Highest amplitudes in the trapped region.
- Exponential decrease on the Right of the BH horizon.
  The **spatial** damping is proportional to the **rate** $\Gamma_a = Im\lambda_a$.

Physical predictions

- At **late times** w.r.t. the **formation** of the BH-WH, i.e. times $\gg 1/Max\Gamma_a$, the mode with the highest Γ_a dominates **all observables**.
  The **classical** and **quantum** descriptions **coincide**.

- At **earlier times**, **if** the ***in*-state** is (near) vacuum, the **quantum settings must be used**, and **all** complex freq. modes contribute to the **observables**

- At **"early" times**, i.e. $\Delta t < T^{\text{Bounce}} = 2\pi/(\omega_a - \omega_{a+1})$
  Hawking radiation as if the **WH were not present.**
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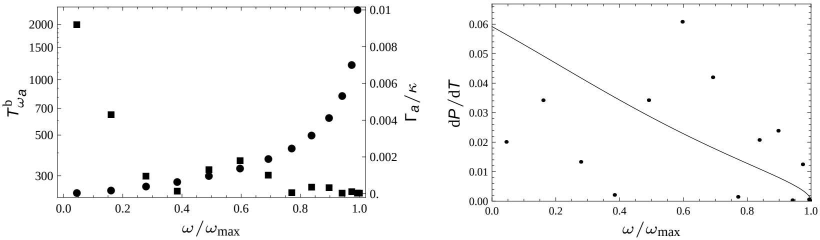
The quantum flux emitted by a BH-WH system, 1

1. A BH-WH system with 13 complex freq. modes.



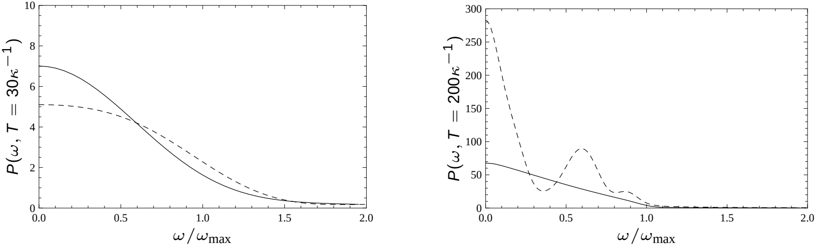
Left: The 13 values of T_a^{Bounce} (dots) and Γ_a (squares)

Right: The **continuous** spectrum obtained **without the WH** vs. the corresponding **discrete** quantity for the **BH-WH** pair.

Very different spectra **in ω-space**.

The flux emitted by a BH-WH system, 2

Fluxes emitted **after a finite lapse of time**
by a single BH (solid line) and the BH-WH pair (dashed).



Left: after $\Delta t = 30/\kappa$, **no** sign yet of discreteness **nor** instab.
the BH-WH pair emits Hawking-like radiation.

Right: after $\Delta t = 200/\kappa$, **discreteness** and **instab.** visible.

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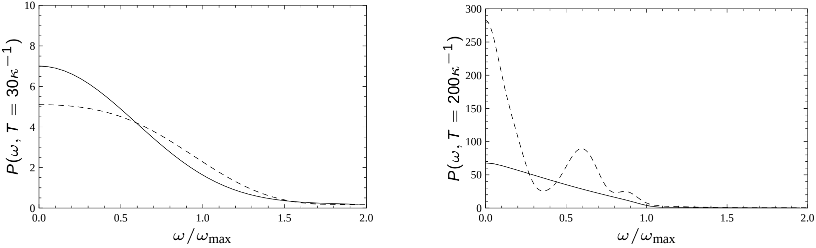

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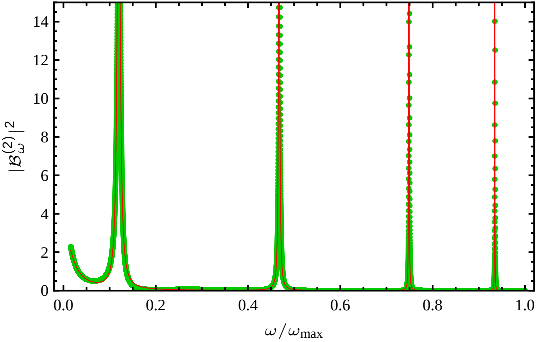
The Technion BH-WH, June 2009, preliminary results



About 4 unstable modes.
Experiment too **short** by a factor of 10 to see the laser effect.
Probably **more** than 4 complex freq. modes.

Classical terms: Induced instability

- When sending a **classical wave** $W_{in}(t, x)$, this **induces** the instability.

- N.B. It does it through the overlaps with the **decaying** modes Z_a

$$b_a \equiv \langle Z_a | W_{in} \rangle \tag{37}$$

which fix the amplitude of the **growing** mode V_a :

$$W_{in}(t, x) \to \sum_a \left[e^{-i\lambda_a t} b_a V_a(x) + p.H.c. \right] . \tag{38}$$

Conclusions

- In flows with **one** sonic B/W horizon, the spectrum
  - is **continuous**, and
  - contains **real** freq., of **both signs** for $\omega < \omega_{\max}$.
  - emitted flux is $\sim$ Hawking radiation when $\omega_{\max} > 3\kappa$.
- In flows with **a pair** of BH-WH horizons, one has
  - a **continuous** spectrum of **real** and **positive** freq., and
  - a **discrete** set of pair of **complex** freq., with $Re\,\lambda_a < \omega_{\max}$.
  - At **late time**, the mode with highest Γ_a dominates all obs.
  - At **early time**, BH-WH flux **as that** from the sole BH.
- When $L\kappa$ suff. **small**,
  no complex freq. modes, hence **no** dyn. instability,
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Extra remarks on Black hole instabilities.

- In 1974, Hawking showed that a Schwarzschild Black Hole **spontaneously** emits thermal radiation.
- Even though it is **microcanonically stable**, it is **canonically unstable**.
- The partition function possesses an unstable $\omega^2 < 0$ **bound mode** (Gross-Perry-Yaffe '82).
- **N.B.** The **same** bound mode is responsible for the **dynamical instability** of 5 dimensional "Black String" (Gregory-Laflamme '93).