Cylindrically and toroidally symmetric solutions with a cosmological constant

J. Podolský and J. B. Griffiths

Institute of Theoretical Physics, Prague

Department of Mathematical Sciences, Loughborough

ERE2010, Granada

September 2010

cylindrically symmetric static vacuum solutions

- for $\Lambda = 0$: Levi-Civita (1919)

$$ds^2 = -\rho^{4\sigma/\Sigma}\,dt^2 + \rho^{-4\sigma(1-2\sigma)/\Sigma}\,dz^2 + C^2\rho^{2(1-2\sigma)/\Sigma}\,d\phi^2 + d\rho^2$$

$$\Sigma = 1 - 2\sigma + 4\sigma^2$$

ρ ... proper radial distance from the axis $\rho = 0$

σ ... mass per unit length on the axis, but only for $\sigma \in (0, \frac{1}{4})$

Minkowski space in cylindrical coordinates when $\sigma = 0$

- for $\Lambda \neq 0$: Linet and Tian (1986)

$$ds^2 = Q^{2/3}\Big(- P^{-2(1-8\sigma+4\sigma^2)/3\Sigma}\,dt^2 + P^{-2(1+4\sigma-8\sigma^2)/3\Sigma}\,dz^2 + C^2 P^{4(1-2\sigma-2\sigma^2)/3\Sigma}\,d\phi^2\Big) + d\rho^2$$

$$Q(\rho) = \frac{1}{\sqrt{3|\Lambda|}}\sinh(\sqrt{3|\Lambda|}\,\rho) \approx \rho$$

for small ρ or Λ: reduces to Levi-Civita

$$P(\rho) = \frac{2}{\sqrt{3|\Lambda|}}\tanh\left(\frac{\sqrt{3|\Lambda|}}{2}\,\rho\right) \approx \rho$$

constant-curvature 3-spaces in cylindrical coordinates

standard metric:

$$\mathrm{d}s^2 = R^2 \left(\frac{\mathrm{d}r^2}{1 - k\,r^2} + r^2(\,\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2) \right)$$

$k = 0,\ +1,\ -1$ for $E^3,\ S^3,\ H^3$ (see the FLRW cosmology)

introducing $\hat\rho$, $\hat z$:

$$\hat\rho = r\ \sin\theta, \qquad \hat z,\ \tan\hat z,\ \tanh\hat z = \frac{r\ \cos\theta}{\sqrt{1 - k\,r^2}}$$

$$\boxed{\mathrm{d}s^2 = R^2 \Big((1 - k\,\hat\rho^2)^{-1}\,\mathrm{d}\hat\rho^2 + (1 - k\,\hat\rho^2)\,\mathrm{d}\hat z^2 + \hat\rho^2\ \mathrm{d}\phi^2 \Big)}$$

<u>for $k = 0$</u>:

$$\mathrm{d}s^2 = R^2 \Big(\mathrm{d}\hat\rho^2 + \mathrm{d}\hat z^2 + \hat\rho^2\ \mathrm{d}\phi^2 \Big)$$

flat space E^3 in cylindrical coordinates: $\hat\rho =$ const. are cylinders $R^1 \times S^1$

$\hat z \in (-\infty, +\infty),\ \ \phi \in [0, 2\pi)$

S^3 in cylindrical-like coordinates

<u>for $k = +1$</u> with $\psi \equiv \hat{z} - \frac{\pi}{2}$:

$$\mathrm{d}s^2 = R^2\Big((1-\hat{\rho}^2)^{-1}\,\mathrm{d}\hat{\rho}^2 + (1-\hat{\rho}^2)\,\mathrm{d}\psi^2 + \hat{\rho}^2\ \mathrm{d}\phi^2\Big)$$

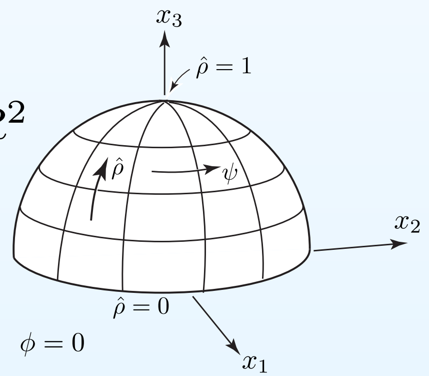
3-sphere S^3 in cylindrical-like coordinates: $\hat{\rho} =$ const. are tori $S^1 \times S^1$

both ψ and ϕ are periodic: $\psi, \phi \in [0, 2\pi)$

two nonintersecting axes at $\hat{\rho} = 0$ and $\hat{\rho} = 1$

indeed: the explicit parametrisation of S^3 as ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = R^2$

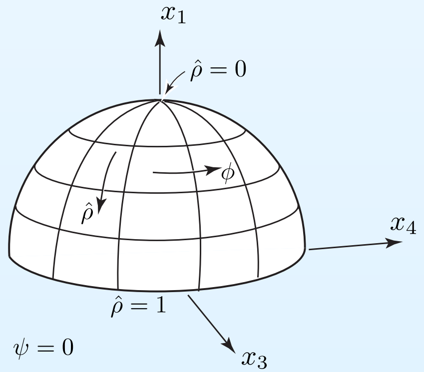
in $\mathrm{d}s^2 = \mathrm{d}{x_1}^2 + \mathrm{d}{x_2}^2 + \mathrm{d}{x_3}^2 + \mathrm{d}{x_4}^2$ is given by



$$x_1 = R\sqrt{1-\hat{\rho}^2}\,\cos\psi \qquad x_3 = R\,\hat{\rho}\,\cos\phi$$
$$x_2 = R\sqrt{1-\hat{\rho}^2}\,\sin\psi \qquad x_4 = R\,\hat{\rho}\,\sin\phi$$

that is

$$x_1^2 + x_2^2 = R^2(1-\hat{\rho}^2) \qquad x_3^2 + x_4^2 = R^2\hat{\rho}^2$$
$$\frac{x_2}{x_1} = \tan\psi \qquad \frac{x_4}{x_3} = \tan\phi$$



ψ and ϕ are “equivalent” angular coordinates in different orientations

interpreting the Linet–Tian vacuum solution with $\Lambda > 0$

in view of the above, we relabel z to ψ and introduce the related conicity parameter B:

$$\mathrm{d}s^2 = Q^{2/3}\Big(- P^{-2(1-8\sigma+4\sigma^2)/3\Sigma}\,\mathrm{d}t^2 + B^2\,P^{-2(1+4\sigma-8\sigma^2)/3\Sigma}\,\mathrm{d}\psi^2 + C^2\,P^{4(1-2\sigma-2\sigma^2)/3\Sigma}\,\mathrm{d}\phi^2\Big) + \mathrm{d}\rho^2$$

$$Q(\rho) = \frac{1}{\sqrt{3\Lambda}}\sin\left(\sqrt{3\Lambda}\,\rho\right) \qquad P(\rho) = \frac{2}{\sqrt{3\Lambda}}\tan\left(\frac{\sqrt{3\Lambda}}{2}\,\rho\right)$$

3 free parameters: σ, B, C

curvature singularities along the axes $\rho = 0$ and $\rho = \frac{\pi}{\sqrt{3\Lambda}}$

with the corresponding deficit angles $2\pi(1-C)$ and $2\pi(1-B)$

each of the two singularities can be removed by replacing the vicinity of the axis by a toroidal part of the dust-filled Einstein static universe (homogeneous, nonsingular) $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \text{“}S^3\text{”}$ in cylindrical-like coordinates with $\hat\rho = \sin\left(\sqrt{\Lambda}\,(\rho-\rho_0)\right)$:

$$\mathrm{d}s^2 = -A_1^2\,\mathrm{d}t^2 + \frac{B_1^2}{\Lambda}\cos^2\left(\sqrt{\Lambda}\,(\rho-\rho_0)\right)\mathrm{d}\psi^2 + \frac{C_1^2}{\Lambda}\sin^2\left(\sqrt{\Lambda}\,(\rho-\rho_0)\right)\mathrm{d}\phi^2 + \mathrm{d}\rho^2$$

matching conditions

we require that the two metrics and their derivatives are continuous across $\rho = \rho_1$

all such conditions can indeed be satisfied by:

$$\cos(\sqrt{3\Lambda}\,\rho_1) = \frac{1 - 8\sigma + 4\sigma^2}{1 - 2\sigma + 4\sigma^2}$$

which uniquely determines ρ_1 in terms of $\sigma \in [0, \frac{1}{4}]$, and

$$\tan^2\left(\sqrt{\Lambda}\,(\rho_1 - \rho_0)\right) = \frac{4\sigma(1-\sigma)}{1 - 4\sigma}$$

$$A_1 = Q(\rho_1)^{-1/3}\, P(\rho_1)^{(1-8\sigma+4\sigma^2)/3\Sigma}$$

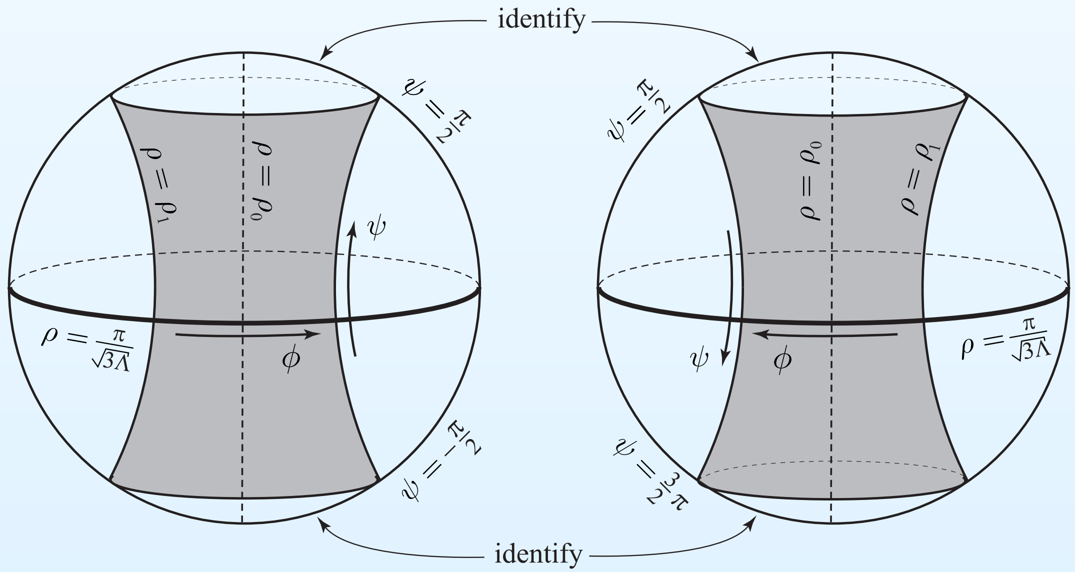
$$B_1 = B\,\Lambda^{-1/2}\, Q(\rho_1)^{-1/3}\, P(\rho_1)^{(1+4\sigma-8\sigma^2)/3\Sigma} \cos\left(\sqrt{\Lambda}(\rho_1 - \rho_0)\right)$$

$$C_1 = C\,\Lambda^{-1/2}\, Q(\rho_1)^{-1/3}\, P(\rho_1)^{-2(1-2\sigma-2\sigma^2)/3\Sigma} \sin\left(\sqrt{\Lambda}(\rho_1 - \rho_0)\right)$$

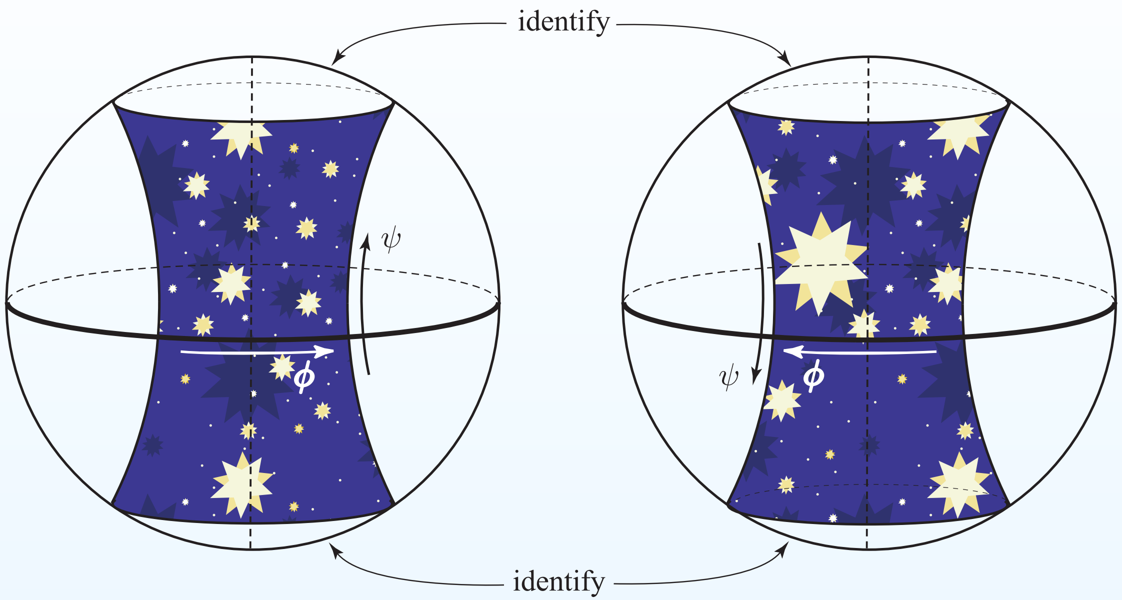
which uniquely determine ρ_0 and A_1, B_1, C_1

resulting spacetime

- curvature singularity at $\rho = 0$ is removed
- it is replaced by the toroidal region $\rho \in [\,\rho_0, \rho_1)$
  which is the uniform Einstein static space filled with dust — the matter source
  (it is regular at the pole ρ_0 when $C_1 = 1$)
- in the external region $\rho \in [\,\rho_1, \frac{\pi}{\sqrt{3\Lambda}})$
  there is the Linet–Tian static vacuum solution
- there remains a curvature singularity at $\rho = \frac{\pi}{\sqrt{3\Lambda}}$
  (alternatively, we can remove this by a toroidal matter source, keeping the singularity at $\rho = 0$)



the mass of the source



the total mass of the toroidal dust matter source is

$$\int_{\rho_0}^{\rho_1}\int_0^{2\pi}\int_0^{2\pi} \mu \sqrt{g_3}\, \mathrm{d}\rho\, \mathrm{d}\psi\, \mathrm{d}\phi = \frac{2\pi B_1 C_1}{\sqrt{\Lambda}}\, \frac{\sigma(1-\sigma)}{(1-4\sigma^2)} \quad \Rightarrow$$

the mass per unit length of the toroid is

$$\frac{\sigma(1-\sigma)}{1-4\sigma^2} \approx \sigma$$

which is consistent with the expectation as $\Lambda \to 0$

the “no source” limit $\sigma = 0$

with

$$p(\rho) = \cos^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}\,\rho\right)$$

the Linet–Tian metric for $\sigma = 0$ reduces to:

$$\mathrm{d}s^2 = p^2(-\,\mathrm{d}t^2 + B^2\ \mathrm{d}\psi^2) + \frac{4C^2}{3\Lambda}\,\frac{(1-p^3)}{p}\ \mathrm{d}\phi^2 + \frac{3}{\Lambda}\,\frac{p}{(1-p^3)}\ \mathrm{d}p^2$$

this is not the (anti-)de Sitter space!

(for Levi-Civita with $\Lambda = 0$ and $\sigma = 0$ we do obtain flat Minkowski space...)

- the spacetime is of type D
- it belongs to the Plebański–Demiański family
- it belongs to the Kundt family
- it is a generalization of the BIII metric

its properties still need to be investigated

extension to higher dimensions

vacuum solution described above can be extended to any higher D-dimensions:

$$\mathrm{d}s^2 = R(\rho)^\alpha \Big(-S(\rho)^{2p_0}\ \mathrm{d}t^2 + \textstyle\sum_{i=1}^{D-2} C_i^{-2} S(\rho)^{2p_i}\ \mathrm{d}\phi_i^2 \Big) + \mathrm{d}\rho^2$$

where $\phi_i \in [0, 2\pi)$, $\qquad R(\rho) = \cos(\beta\rho), \qquad S(\rho) = \beta^{-1}\tan(\beta\rho),$

$$\alpha = \frac{4}{D-1}, \qquad \beta = \sqrt{\frac{(D-1)\Lambda}{2(D-2)}},$$

C_i are corresponding conicity parameters and the constants p_i satisfy

$$\sum_{i=0}^{D-2} p_i = 1, \qquad \sum_{i=0}^{D-2} p_i^2 = 1$$

- for $D = 4$ this reduces to the Linet–Tian solution with the parameters
$$p_0 = \frac{2\sigma}{\Sigma}, \qquad p_1 = -\frac{2(1-2\sigma)\sigma}{\Sigma}, \qquad p_2 = \frac{1-2\sigma}{\Sigma}$$
- the $\Lambda < 0$ counterpart has been recently given by Sarıoğlu and Tekin (2009)

for more information and references see

J. B. Griffiths and J. Podolský,

The Linet–Tian solution with a positive cosmological constant in four and higher dimensions,

Phys. Rev. D **81** 064015 (2010)