The behaviour of geodesics in constant-curvature spacetimes with expanding impulsive gravitational waves

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Background spacetimes

Spacetimes of constant curvature $R = 4\Lambda$

According to the signum of Λ we distinguish:

- *Minkowski spacetime:* $\Lambda = 0$
- *de Sitter spacetime:* $\Lambda > 0$
- *anti-de Sitter spacetime:* $\Lambda < 0$

We can write these spacetimes in the conformally flat form, namely

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\left[1 + \frac{\Lambda}{12}(-t^2 + x^2 + y^2 + z^2)\right]^2} \, ,$$

or equivalently

$$\mathrm{d}s^2 = \frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - 2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V}}{[1 + \frac{1}{6}\Lambda(\zeta\bar{\zeta} - \mathcal{U}\mathcal{V})]^2} \, .$$

Background spacetimes

(anti-)de Sitter universe

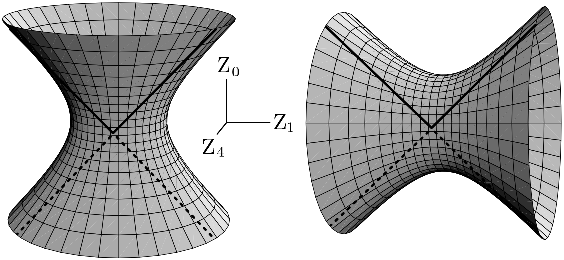
(Anti-)de Sitter spacetime is visualized as a *4-dimensional hyperboloid*

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + \sigma Z_4^2 = \sigma a^2 \ ,$$

in flat 5-dimensional space

$$\mathrm{d}s^2 = -\mathrm{d}{Z_0}^2 + \mathrm{d}{Z_1}^2 + \mathrm{d}{Z_2}^2 + \mathrm{d}{Z_3}^2 + \sigma \mathrm{d}{Z_4}^2 \ ,$$

where $a = \sqrt{\frac{3}{|\Lambda|}}$, and $\sigma = \mathrm{sign}\,\Lambda$.



Impulsive spherical gravitational waves

Expanding impulsive spherical gravitational waves can be constructed using the *"cut and paste" method* [Penrose (1972)], i. e. "cutting" a spacetime along a null cone and "putting it together" with a suitable warp:

$$\left[Z, \bar{Z}, V, U = 0_-\right]_{\mathcal{M}_-} \equiv \left[h(Z), \bar{h}(\bar{Z}), \frac{(1+\epsilon h\bar{h})V}{(1+\epsilon Z\bar{Z})|h'|}, U = 0_+\right]_{\mathcal{M}_+} .$$

The impulse is located on the hypersurface $U = 0$, which is a sphere expanding with the speed of light ($x^2 + y^2 + z^2 = t^2$).

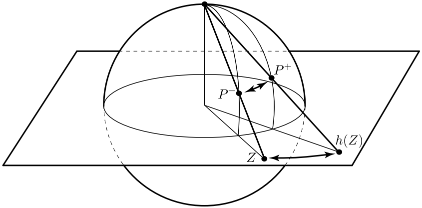


Figure: Geometrical interpretation of the Penrose junction conditions.

Continuous line element

Continuous metric form which describes such waves is [see Hogan (1992, 1993, 1994), Podolský and Griffiths (1999)]

$$\mathrm{d}s^2 = \frac{2\left|\frac{V}{p}\mathrm{d}Z + U\Theta(U)p\bar{H}\mathrm{d}\bar{Z}\right|^2 + 2\mathrm{d}U\mathrm{d}V - 2\epsilon\mathrm{d}U^2}{\left[1+\frac{1}{6}\Lambda U(V-\epsilon U)\right]^2} \ ,$$

where $\quad p = 1 + \epsilon Z\bar{Z} \ , \qquad \epsilon = -1, 0, +1 \ .$

Relation of this line element to the pure background metric is

- *behind the impulse*: $U < 0$

$$\mathcal{V}^- = \frac{V}{p} - \epsilon U \ , \qquad \mathcal{U}^- = \frac{Z\bar{Z}}{p}V - U \ , \qquad \zeta^- = \frac{Z}{p}V \ ,$$

- *in front of the impulse*: $U > 0$

$$\mathcal{V}^+ = AV - DU \ , \qquad \mathcal{U}^+ = BV - EU \ , \qquad \zeta^+ = CV - FU \ ,$$

where the parameters A, B, C, D, E a F are functions of Z resp. $\bar{Z}$.

Refraction formulae in conformally flat coordinates

Suppose C^1 geodesics $Z = Z(\tau)$, $U = U(\tau)$ and $V = V(\tau)$.
Denote the positions and velocities in the *interaction time* τ_i as

$$Z_i = Z(\tau_i)\ , \quad V_i = V(\tau_i)\ , \quad \dot{U}_i = \dot{U}(\tau_i)\ , \quad \dot{V}_i = \dot{V}(\tau_i)\ , \quad \dot{Z}_i = \dot{Z}(\tau_i)\ .$$

Apply transformations *in front of* and *behind* the impulse and express

$$x_i^-(Z_i, V_i)\ , \qquad \dot{x}_i^-(Z_i, V_i, \dot{U}_i, \dot{V}_i, \dot{Z}_i)\ ,$$

etc. for $y_i^-,\ \dot{y}_i^-,\ z_i^-,\ \dot{z}_i^-,\ t_i^-,\ \dot{t}_i^-$,

and

$$Z_i(x_i^+, y_i^+, z_i^+, t_i^+)\ , \qquad \dot{U}_i(x_i^+, y_i^+, z_i^+, t_i^+, \dot{x}_i^+, \dot{y}_i^+, \dot{z}_i^+, \dot{t}_i^+)\ ,$$

$$V_i(x_i^+, y_i^+, z_i^+, t_i^+)\ , \qquad \dot{V}_i(x_i^+, y_i^+, z_i^+, t_i^+, \dot{x}_i^+, \dot{y}_i^+, \dot{z}_i^+, \dot{t}_i^+)\ ,$$

$$\dot{Z}_i(x_i^+, y_i^+, z_i^+, t_i^+, \dot{x}_i^+, \dot{y}_i^+, \dot{z}_i^+, \dot{t}_i^+)\ .$$

Write the interaction parameters behind the impulse as functions of the parameters in front of the impulse. The result is

- for *positions*

$$x_i^- = |h'|\frac{Z_i + \bar{Z}_i}{h + \bar{h}} x_i^+ \,, \qquad y_i^- = |h'|\frac{Z_i - \bar{Z}_i}{h - \bar{h}} y_i^+ \,,$$

$$z_i^- = |h'|\frac{Z_i\bar{Z}_i - 1}{|h|^2 - 1} z_i^+ \,, \qquad t_i^- = |h'|\frac{Z_i\bar{Z}_i + 1}{|h|^2 + 1} t_i^+ \,,$$

- for *velocities*

$$\begin{aligned}
\dot{x}_i^- &= a_x \dot{x}_i^+ + b_x \dot{y}_i^+ + c_x \dot{z}_i^+ + d_x \dot{t}_i^+ \,, \\
\dot{y}_i^- &= a_y \dot{x}_i^+ + b_y \dot{y}_i^+ + c_y \dot{z}_i^+ + d_y \dot{t}_i^+ \,, \\
\dot{z}_i^- &= a_z \dot{x}_i^+ + b_z \dot{y}_i^+ + c_z \dot{z}_i^+ + d_z \dot{t}_i^+ \,, \\
\dot{t}_i^- &= a_t \dot{x}_i^+ + b_t \dot{y}_i^+ + c_t \dot{z}_i^+ + d_t \dot{t}_i^+ \,,
\end{aligned}$$

where the coefficients a, b, c, d are complicated functions of Z_i and $h(Z_i)$.

Refraction formulae

Using angles characterizing positions and velocity directions

Define angles in (x, z) and (y, z) planes

- $\alpha^{\pm}$ and $\gamma^{\pm}$ describe *position of the particle*
- $\beta^{\pm}$ and $\delta^{\pm}$ describe *inclination of the velocity vector*

$$\tan \alpha^{\pm} = \frac{x_i^{\pm}}{z_i^{\pm}} , \qquad \tan \beta^{\pm} = \frac{\dot{x}_i^{\pm}}{\dot{z}_i^{\pm}} .$$

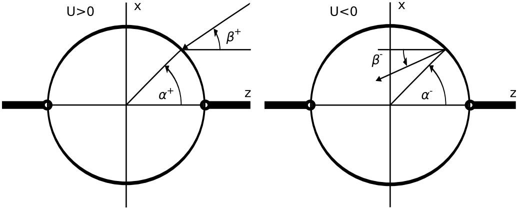


Figure: Geometrical meaning of angles in (x, z) plane. Superscript $+$ denotes quantities in front of the impulse, and $-$ behind the impulse.

Then, the formulae for positions and veocities can be rewritten in the form

$$\cot\alpha^{-} = \frac{(|Z_i|^2-1)\,\mathrm{Re}h}{\mathrm{Re}Z_i\,(|h|^2-1)}\cot\alpha^{+}\ ,$$

$$\cot\gamma^{-} = \frac{(|Z_i|^2-1)\,\mathrm{Im}h}{\mathrm{Im}Z_i\,(|h|^2-1)}\cot\gamma^{+}\ ,$$

and

$$\tan\beta^{-} = \frac{v_z^{+}(a_x\tan\beta^{+}+b_x\tan\delta^{+}+c_x)+d_x}{v_z^{+}(a_z\tan\beta^{+}+b_z\tan\delta^{+}+c_z)+d_z}\ ,$$

$$\tan\delta^{-} = \frac{v_z^{+}(a_y\tan\beta^{+}+b_y\tan\delta^{+}+c_y)+d_y}{v_z^{+}(a_z\tan\beta^{+}+b_z\tan\delta^{+}+c_z)+d_z}\ ,$$

where we introduced velocities with respect to the frame

$$(v_x^{\pm}, v_y^{\pm}, v_z^{\pm}) = \left(\frac{\dot{x}_i^{\pm}}{\dot{t}_i^{\pm}}, \frac{\dot{y}_i^{\pm}}{\dot{t}_i^{\pm}}, \frac{\dot{z}_i^{\pm}}{\dot{t}_i^{\pm}}\right)\ .$$

Example

Impulsive spherical wave generated by a snapping cosmic string

The *complex mapping* is $h(Z) = Z^{1-\delta}$ where δ *characterizes deficit angle* given by the presence of cosmic string outside the impulse

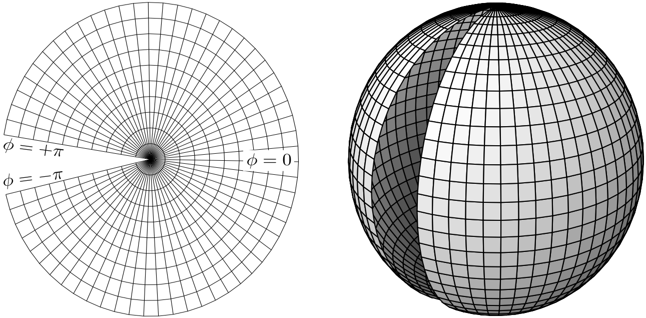


Figure: Mapping $Z \to h(Z) = Z^{1-\delta}$ in the complex plane (on the left) coresponds to a wedge in the Riemann sphere (on the right). The deficit angle is $2\pi\delta$.

Example

The ring of test particles standing in front of the impulse

- wave is generated by the snapping string: $h(Z) = Z^{1-\delta}$
- particles are standing in (x^+, z^+) plane

Resulting motion remains only in (x, z) plane with relevant coefficients in the velocity transformation

$$d_x = \frac{-\delta(1-\frac{\delta}{2})}{2(1-\delta)} \left(Z_i^{1-\delta} + Z_i^{\delta-1} \right) ,$$

$$d_z = \frac{1}{2(1-\delta)} \left[\left(1 - \frac{\delta}{2}\right)^2 \left(Z_i^{\delta} - Z_i^{-\delta} \right) + \frac{\delta^2}{4} \left(Z_i^{2-\delta} - Z_i^{\delta-2} \right) \right] .$$

Therefore, the changes of angles are

$$\cot\alpha^- = Z_i^{-\delta} \frac{Z_i^2 - 1}{Z_i^{2-2\delta} - 1} \cot\alpha^+ ,$$

$$\tan\beta^- = \frac{-\delta(1-\frac{\delta}{2}) \left(Z_i^{1-\delta} + Z_i^{\delta-1} \right)}{\left(1 - \frac{\delta}{2}\right)^2 \left(Z_i^{\delta} - Z_i^{-\delta} \right) + \frac{\delta^2}{4} \left(Z_i^{2-\delta} - Z_i^{\delta-2} \right)} .$$

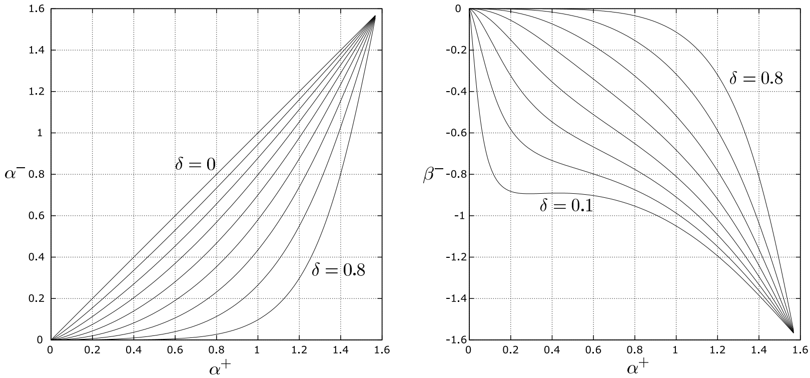


Figure: Shift of the position of particle induced by the wave: $\alpha^-(\alpha^+)$ (on the left). Dependence of velocity vector inclination on particle's position in front of impulse: $\beta^-(\alpha^+)$ (on the right). The curves corespond to different values of the deficit angle.

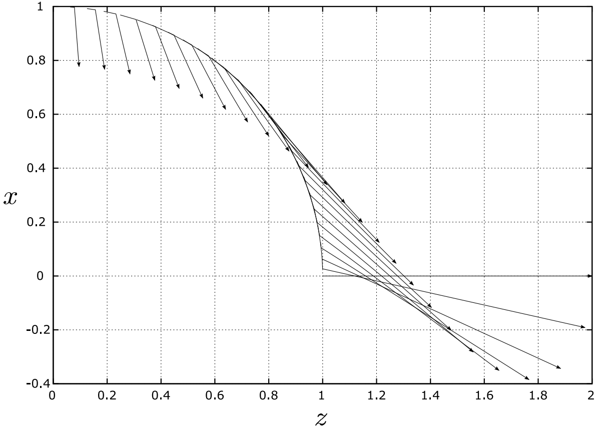


Figure: The effect of the impulse with $\delta = 0.2$ on a ring of initially static test particles in the (x, z) plane. The impulse is scaled here in such a way that it is given by a unit sphere on both sides of the impulse.

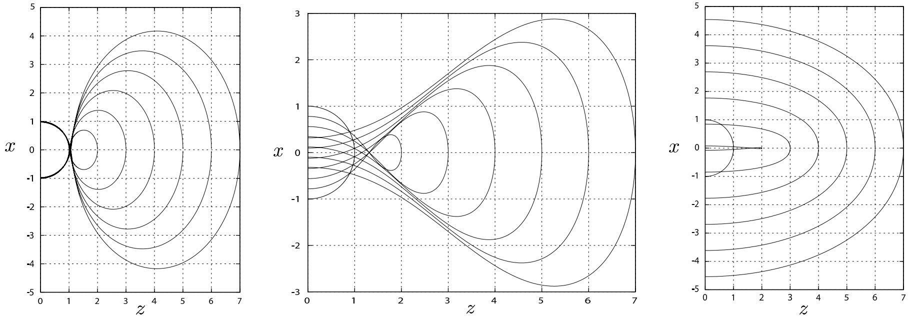


Figure: Time sequence showing the deformation of the ring of test particles (indicated here by an initial semi-circle of unit radius for $\alpha^+ \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) caused by the spherical impulse generated by a snapping cosmic string with $\delta = 0.005$ (left), $\delta = 0.2$ (middle) and $\delta = 0.8$ (right).

Alternative form of the refraction formulae for $\Lambda \neq 0$

In 5-dimensional representation of (anti-)de Sitter spacetime we obtain

$$Z_{0i}^- = |h'| \frac{Z_i \bar{Z}_i + 1}{|h|^2 + 1} Z_{0i}^+ \,, \qquad Z_{1i}^- = |h'| \frac{Z_i \bar{Z}_i - 1}{|h|^2 - 1} Z_{1i}^+ \,,$$

$$Z_{2i}^- = |h'| \frac{Z_i + \bar{Z}_i}{h + \bar{h}} Z_{2i}^+ \,, \qquad Z_{3i}^- = |h'| \frac{Z_i - \bar{Z}_i}{h - \bar{h}} Z_{3i}^+ \,,$$

$$Z_{4i}^- = a = Z_{4i}^+ \,,$$

and

$$\begin{pmatrix} \dot{Z}_{2i}^- \\ \dot{Z}_{3i}^- \\ \dot{Z}_{1i}^- \\ \dot{Z}_{0i}^- \\ \dot{Z}_{4i}^- \end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x & d_x & \omega_x \\ a_y & b_y & c_y & d_y & \omega_y \\ a_z & b_z & c_z & d_z & \omega_z \\ a_t & b_t & c_t & d_t & \omega_t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{Z}_{2i}^+ \\ \dot{Z}_{3i}^+ \\ \dot{Z}_{1i}^+ \\ \dot{Z}_{0i}^+ \\ \dot{Z}_{4i}^+ \end{pmatrix} ,$$

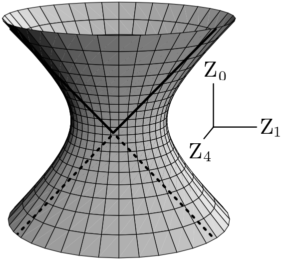
where $\omega_j = -\frac{1}{2a} \left(a_j Z_{2i}^+ + b_j Z_{3i}^+ + c_j Z_{1i}^+ + d_j Z_{0i}^+ - Z_{ji}^- \right)$ and $j = t, z, x, y$.

Example

Comoving particles in de Sitter spacetime

The global parametrization of de Sitter spacetime is

$$Z_0 = a \sinh \frac{t}{a} \ ,$$
$$Z_1 = a \cosh \frac{t}{a} \sin\chi \cos\theta \ ,$$
$$Z_2 = a \cosh \frac{t}{a} \sin\chi \sin\theta \cos\phi \ ,$$
$$Z_3 = a \cosh \frac{t}{a} \sin\chi \sin\theta \sin\phi \ ,$$
$$Z_4 = a \cosh \frac{t}{a} \cos\chi \ .$$



Apply previous equations to comoving particles in this parametrization, i.e.

$$\chi^+ = const. \ , \qquad \theta^+ = const. \ , \qquad \phi^+ = const. \ .$$

These worldlines are timelike geodesics.

Example

Location of the impulse

Impulse ($U = 0$) in the de Sitter background corresponds to $Z_4 = a$,

$$ {Z_1}^2 + {Z_2}^2 + {Z_3}^2 = {Z_0}^2 \ , \qquad \text{or} \qquad \cosh\frac{t}{a}\cos\chi = 1 \ . $$

Location of cosmic string is given by the condition $Z_{23}^+ = 0$, i.e.

$$ Z_{23}^+ = \frac{Z_2^+}{\cos\phi^+} = \frac{Z_3^+}{\sin\phi^+} = a\cosh\frac{t^+}{a}\sin\chi^+\sin\theta^+ = 0 \ . $$

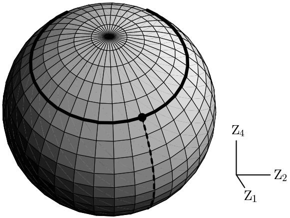


Figure: Cosmic string and the impulse in de Sitter space.

Example

Effect of the impulse on comoving particles in de Sitter space

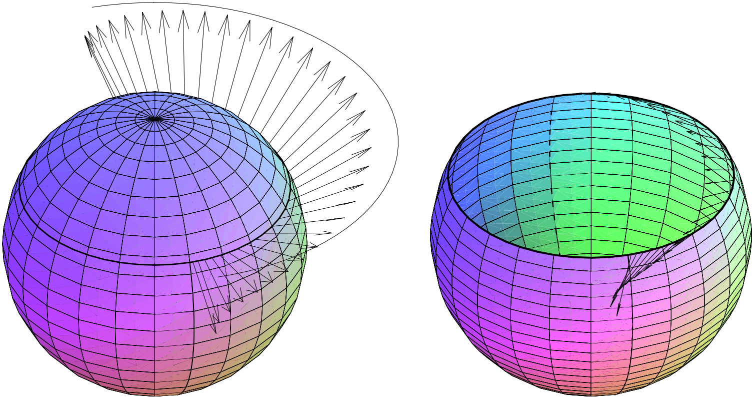


Figure: 5-dimensional velocity vectors of the initially comoving particles behind the impulse (on the left). Velocity vectors of the same particles behind the impulse, with substracted comoving part in front of the impulse (on the right). De Sitter space is scaled on the unit sphere.

Conclusions

- complete description of the influence of expanding spherical impulsive gravitational waves on free test particles in spacetimes of constant curvature
- generalization of previous results for Minkowski to any cosmological constant Λ (de Sitter, anti-de Sitter universe)
- derivation of general refraction formulae describing shift of positions and change of velocity vectors of these particles
- investigation and visualization of the effect of the impulsive wave generated by a snapping cosmic string

For more detail see:

Podolský J., Švarc R.: *Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant*, Phys. Rev. D **81** (2010) 124035.

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- Podolský J., Steinbauer R.: *Geodesics in spacetimes with expanding impulsive gravitational waves*, Phys. Rev. D **67** (2003) 064013.