Symmetric solutions of Einstein's equations in higher dimensions

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Metrics in dimension $D > 4$

- Black holes: Myers-Perry, Emparan-Reall and other (see Livingreview by Emparan and Reall)
- Kaluza-Klein configurations: monopoles, Gross-Perry metrics
- Metrics with D-2 abelian symmetries (solution generation techniques applicable, see review by Harmark)
- Algebraically special metrics (the generalized Robinson-Trautman class, the Kundt class, see review by Coley)

Our goal

Vacuum metrics with symmetry group $SO(n+1)$ acting on n-dimensional orbits

Plan

- Reduction of the Einstein equations via SO(n + 1) symmetry
- Further reduction via a diferential ansatz
- Generation of new solutions based on the Gross-Perry metrics

Jakimowicz M. and Tafel J. 2008, SO(n + 1) symmetric solutions of the Einstein equations in higher dimensions, *Class. Quantum Grav.*, **25** 175002

Jakimowicz M. and Tafel J. 2009, Generalization of the Gross-Perry Metrics, *Int. J. Theor. Phys.* **48** 2876

Dimensional reduction of the Einstein equations

Let $SO(n+1)$ acts as an isometry group on n-dimensional spheres S_n, $n > 1$.

Spacetime metric

$$g = g_{ab}dx^a dx^b - e^{2f} d\Omega_n^2$$

$d\Omega_n^2$ is the standard metric of S_n

g_{ab} and f are functions of coordinates x^a

$a, b = 0, 1, \cdots, N$

$N = 1$

The Schwarzschild-Tangerhlini metric

$$g = \left(1 - \frac{2M}{r^{n-1}}\right) dt^2 - \left(1 - \frac{2M}{r^{n-1}}\right)^{-1} dr^2 - r^2 d\Omega_n^2$$

$N > 1$

$$\widetilde{g}_{ab} = e^{\frac{2n}{N-1}f} g_{ab}$$

$$\phi = \sqrt{\frac{n\left(n+N-1\right)}{\left(N-1\right)}} f$$

Equations for $\widetilde{g}$ and ϕ

$$\widetilde{R}_{ab} - \frac{1}{2}\widetilde{g}_{ab} = \phi_{;a}\phi_{;b} + \left(V(\phi) - \frac{1}{2}\phi^{;c}\phi_{;c}\right)\widetilde{g}_{ab}$$

$$\phi^{;a}{}_{;a} = -V_{,\phi}$$

where

$$V = -\frac{1}{2}n\left(n-1\right)e^{-2\sqrt{\frac{n+N-1}{n(N-1)}}\phi}$$

Further reduction

Assumptions:

- The normal vector field of surfaces $\phi = const$ is spacelike and geodesic
- The surfaces are Einstein spaces

Form of the metric

$$\tilde{g} = \tilde{g}_{ij} dx^i dx^j - ds^2 \ , \quad i = 0, \ldots, N-1$$

$$\phi = \phi(s)$$

$$\tilde{g}_{ij} = \beta^{\frac{2}{N}} (\gamma e^{P\tau})_{ij}$$

β and τ are functions of s

γ_{ij} (of the Lorentz signature) and P_{ij} are symmetric tensors depending on x^i

Conditions for P_{ij} and γ_{ij}

$$P^i{}_i = 0 \ , \qquad P^i{}_j P^j{}_i = 2c = const$$

$$P^k{}_{i;k} = 0$$

$$R^i{}_j(\gamma e^{P\tau}) = \lambda \delta^i{}_j \ , \quad \lambda = const$$

Example of γ and P for any $N > 1$

$$\gamma_{ij} = \mathrm{diag}(+1, -1, -1, ..) \ , \quad P_{ij} = P_{ji} = const \ , \quad P^i{}_i = 0 \ , \ \lambda = 0$$

Example of γ and P for $N = 2$

$$(\gamma e^{P\tau})_{ij} dx^i dx^j = \frac{du dv}{(1 + \frac{\lambda}{4} uv)^2} + \tau h(u) du^2 \ , \ h = h(u)$$

Equations for $\beta(s)$, $\phi(s)$ and $\tau(s)$

$$(\beta\dot{\phi})^{\dot{}} = -\beta V_{,\phi}$$

$$(1-\frac{1}{N})\frac{\dot{\beta}^2}{\beta^2} - \frac{2c}{\beta^2} - \dot{\phi}^2 + 2V = -N\lambda\beta^{-\frac{2}{N}}$$

$$\beta\dot{\tau} = 2$$

Conclusion

$$g = e^{-2\sqrt{\frac{n}{(N-1)(n+N-1)}}\phi}\left(\beta^{\frac{2}{N}}(\gamma e^{P\tau})_{ij}dx^i dx^j - ds^2\right) - e^{2\sqrt{\frac{N-1}{n(n+N-1)}}\phi}d\Omega_n^2$$

is an $(N+n+1)$–dimensional vacuum metric

Examples of metrics

- Generalized pp-wave of type N

$$g = du\left(dv - r^{1-n}h(u)du\right) - dr^2 - r^2 d\Omega_n^2$$

- The Kundt metric of type II with nonconstant scalar invariants

$$g = du\left(\frac{4r^2 dv}{(n+1)(1-uv)^2} - r^{1-n}h(u)du\right) - dr^2 - \frac{n-1}{n+1}r^2 d\Omega_n^2$$

Comments

- Equations for $\beta(s)$ and $\phi(s)$ depend on tensors γ and P via constant c only.
- Some solutions can be obtained from known metrics and then implemented to other forms of γ and P.

Solutions related to the Gross-Perry metrics

Let $s = s(r)$ and $\alpha = s_{,r}$.

$$\alpha = \pm \frac{1}{n-1} |r|^{-l-1} |r - r_0|^{l-p} |r + r_0|^{l+p}$$

$$\beta = (r^2 - r_0^2)\alpha \ , \qquad e^{\sqrt{\frac{n+N-1}{n(N-1)}}\phi} = (n-1)|r\alpha|$$

$$\tau = r_0^{-1} \ln \left| \frac{r + r_0}{r - r_0} \right|$$

p and $r_0 > 0$ are free parameters

$$l = \frac{n+N-1}{(n-1)(N-1)}$$

$$c = 2r_0^2 \left[\frac{n}{n-1} - p^2 \frac{(n-1)(N-1)^2}{N(n+N-1)} \right]$$

Metrics for $N = 2$, P=const and $c > 0$

$$g = \left|\frac{r - r_0}{r + r_0}\right|^{p' - q} dt^2 - \left|\frac{r - r_0}{r + r_0}\right|^{p' + q} dy^2$$

$$- \frac{|r + r_0|^{\frac{2p' + 2}{n-1}}}{|r|^{\frac{2n}{n-1}} |r - r_0|^{\frac{2p' - 2}{n-1}}} \left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right)$$

Parameters p' and q are constrained by

$$(n+1)p'^2 + (n-1)q^2 = 2n$$

$n = 2$ yields the Gross-Perry metric

Properties of the metrics

- Additional Killing vectors ∂_t and ∂_y
- In the limit $r \to \infty$ the metrics tend to the flat metric

$$dt^2 - dy^2 - dr'^2 - r'^2 d\Omega_n^2 \ , \quad r' = r^{\frac{1}{n-1}}$$

- Algebraic type I
- Coordinate singularity at $r = 0$ and essential singularities at $r = \pm r_0$
- In the Kaluza-Klein approach the metrics split into
  - asymptotically flat metric on $y = const$
  - asymptotically constant scalar field

Metrics for $N = 2$, P=const and $c < 0$

$$g = \left|\frac{r-r_0}{r+r_0}\right|^{p'}\left[\cos\left(q\ln\left|\frac{r+r_0}{r-r_0}\right|\right)(dt^2-dy^2)+2\sin\left(q\ln\left|\frac{r+r_0}{r-r_0}\right|\right)dtdy\right]$$

$$-\frac{|r+r_0|^{\frac{2p'+2}{n-1}}}{|r|^{\frac{2n}{n-1}}|r-r_0|^{\frac{2p'-2}{n-1}}}\left(\frac{dr^2}{(n-1)^2}+r^2d\Omega_n^2\right)$$

$$(n+1)p'^2-(n-1)q^2=2n$$

Change of properties with respect to $c > 0$

- stationarity instead of staticity
- for some range of parameters regions $r = \pm r_0$ are at infinite distance (the Riemann tensor vanishes there but the metric degenerates)
- nonvanishing electromagnetic field in the Kaluza-Klein approach

Metrics for $N = 2$, $P \neq$const and $c = 0$

$$g = \left|\frac{r - r_0}{r + r_0}\right|^{\pm\sqrt{\frac{2n}{n+1}}} \left(dudv + \ln\left|\frac{r + r_0}{r - r_0}\right| h(u)du^2\right)$$

$$- \frac{|r + r_0|^{\frac{2}{n-1}(\pm\sqrt{\frac{2n}{n+1}}+1)}}{|r|^{\frac{2n}{n-1}}|r - r_0|^{\frac{2}{n-1}(\pm\sqrt{\frac{2n}{n+1}}-1)}} \left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2\right)$$

- Additional null Killing vector field ∂_v
- The metrics belong to generalized Kundt's class
- In the limit $r \to \infty$ they tend to the flat metric

$$dudv - dr'^2 - r'^2 d\Omega_n^2\ , \quad r' = r^{\frac{1}{n-1}}$$

- Algebraic type II_i for $h \neq 0$ and type D for $h = 0$
- Coordinate singularity at $r = 0$ and essential singularities at $r = \pm r_0$

Summary of results

- A method of solving the Einstein equations in higher dimensions
- Generalization of the Gross-Perry metrics
- New classes of solutions (nonstatic and the generalized Kundt metrics)
- Generation of Kaluza-Klein monopoles and dyons via symmetries

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