# Discretisation parameter and operator ordering in loop quantum cosmology with the cosmological constant


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# Loop Quantum Cosmology (LQC)

**LQC is based on Loop Quantum Gravity**

- Canonical Quantization of GR
- Canonical variables : Ashtekar connection ($A=\Gamma+\gamma K$)

  Triad

Γ : Spin connection
K : Extrinsic curvature
γ : Immirzi parameter

**LQC is based on symmetry reduction**

ex) flat FRW spacetime $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$

- Triad : $E^a_i = pV_0^{-(2/3)}\sqrt{\bar{q}}\bar{e}^a_i$ $\quad |p| = V_0^{2/3}a^2$
- Ashtekar connection : $A^i_a = cV_0^{-(1/3)}\bar{\omega}^i_a$ $\quad c = V_0^{1/3}\mathrm{sgn}(p)\gamma\dot{a}$

- Poisson bracket : $\{c, p\} = 8\pi G\gamma/3$

# Hamiltonian constraint

In flat FRW model

Hamiltonian (gravitational part+cosmological constant)

$$C_{\rm grav} = \frac{1}{16\pi G\gamma^2}\int_\nu {\rm d}^3x \left( -\frac{1}{\sqrt{|{\rm det}E^a_i|}}\epsilon_{ijk}F^i_{ab}E^{aj}E^{bk} + 2\gamma^2\sqrt{|{\rm det}E^a_i|}\Lambda \right)$$

To quantise the Hamiltonian,
we rewrite the Hamiltonian in terms of holonomy and flux

holonomy $h_i^{(\lambda)} = \mathcal{P}e^{\int_e A^i_a \tau_i {\rm d}x^a} = e^{\lambda c \tau_i}$

$\lambda$ is the length of holonomy

flux $\mathcal{E}[\mathcal{S}, f] = \int_{\mathcal{S}} {\rm d}^2\sigma n_a E^a_i f^i \propto p$

$$C_{\rm grav} = \frac{1}{16\pi G\gamma^2}\left( -\frac{4{\rm sgn}(p)}{8\pi\lambda^3 G\gamma}\sum_{ijk}\epsilon^{ijk}{\rm Tr}\left[ \underbrace{h_i^{(\lambda)}h_j^{(\lambda)}\left(h_i^{(\lambda)}\right)^{-1}\left(h_j^{(\lambda)}\right)^{-1}}_{(=F)} \underbrace{h_k^{(\lambda)}\left\{\left(h_k^{(\lambda)}\right)^{-1}, V\right\}}_{(=EE/\sqrt{{\rm det}E})}\right] + 2\gamma^2\Lambda V \right)$$

# Quantisation

An orthonormal basis {|μ>} for kinematical Hilbert space is given by a set of eigenstate of $\hat{p}$

$$\widehat{p}|\mu\rangle = \frac{8\pi\gamma l_{\rm Pl}^2}{6}\mu|\mu\rangle$$

The states |μ> are also eigenstates of volume operator

$$\widehat{V}|\mu\rangle = \widehat{|p|^{3/2}}|\mu\rangle = V_\mu|\mu\rangle$$

**Quantum Hamiltonian constraint** $\widehat{C}_{\rm grav}|\Psi\rangle = 0$

$$\widehat{C}_{\rm grav} = \frac{1}{16\pi l_{\rm Pl}^2\gamma^2}\left(\frac{96i\,({\rm sgn}(p))}{8\pi\gamma l_{\rm Pl}^2}\widehat{\frac{1}{\lambda^3}}\widehat{\sin^2\frac{\lambda c}{2}\cos^2}\frac{\lambda c}{2}\left[\sin\frac{\lambda c}{2}V\widehat{\cos\frac{\lambda c}{2} - \cos}\frac{\lambda c}{2}V\sin\frac{\lambda c}{2}\right] + 2\gamma^2\Lambda\widehat{V}\right)$$

# Hamiltonian constraint

- $\widehat{C}_{\rm grav}|\Psi\rangle = 0$

$$\widehat{C}_{\rm grav} = \frac{1}{16\pi l_{\rm Pl}^2\gamma^2}\left(\frac{96i\,(\mathrm{sgn}(p))}{8\pi\gamma l_{\rm Pl}^2}\widehat{\frac{1}{\lambda^3}}\,\widehat{\sin^2\frac{\lambda c}{2}\cos^2\frac{\lambda c}{2}}\,\widehat{\left[\sin\frac{\lambda c}{2}V\cos\frac{\lambda c}{2}-\cos\frac{\lambda c}{2}V\sin\frac{\lambda c}{2}\right]}+2\gamma^2\Lambda\widehat{V}\right)$$

Quantum ambiguities
- λ is constant or non-constant
- operator orderings

**To-do**

- λ is constant or non-constant
  → We analyse the behaviour of the wave function in the large volume limit

cf.) Nelson, Sakellariadou (2007)

- operator orderings
  → We analyse whether or not the initial singularity is avoided

cf.) Bojowald (2001)

# Equi-area discretisation

Ashtekar, Bojowald, Lewandoeski (2003)

$\lambda = \mu_0 = const.$

basic operator : $\widehat{e^{i\mu_0 c/2}}|\mu\rangle = |\mu + \mu_0\rangle$

· $\widehat{C}_{\rm grav}|\Psi\rangle = 0$

$$\widehat{C}_{\rm grav} = \frac{1}{16\pi l_{\rm Pl}^2 \gamma^2}\left(\frac{96i\,(\mathrm{sgn}(p))}{8\pi\gamma l_{\rm Pl}^2 \mu_0^3}\widehat{F}\widehat{EE} + 2\gamma^2 \Lambda \widehat{V}\right)$$

$$\left(\widehat{F} = \widehat{\sin^2\frac{\mu_0 c}{2}\cos^2\frac{\mu_0 c}{2}}, \quad \widehat{EE} = \sin\frac{\mu_0 c}{2} V \cos\widehat{\frac{\mu_0 c}{2} - \cos}\frac{\mu_0 c}{2} V \sin\frac{\mu_0 c}{2}\right)$$

We expand states |ψ> in eigenstates |μ>

$\Psi(\mu) = \langle\Psi|\mu\rangle$ and obtain the difference equation

$$|V_{\mu+5\mu_0} - V_{\mu+3\mu_0}|\,\Psi(\mu + 4\mu_0) - \left(2\,|V_{\mu+\mu_0} - V_{\mu-\mu_0}| - \frac{16\pi\gamma^3 l_{\rm Pl}^2 \mu_0^3}{3}\Lambda V_\mu\right)\Psi(\mu) + |V_{\mu-3\mu_0} - V_{\mu-5\mu_0}|\,\Psi(\mu - 4\mu_0) = 0$$

# Equi-volume discretisation

Ashtekar, Pawlowski, Singh (2006)

$$\lambda = \bar{\mu} = \frac{3\sqrt{3}}{2}|p|^{-1/2}$$

basic operator : $e^{i\bar{\mu}c/2}|\mu\rangle = ???$

$$\longrightarrow \exp(i\bar{\mu}\hat{c}/2)|v\rangle = |v+1\rangle$$

$$v = K\mathrm{sgn}(\mu)|\mu|^{3/2}$$

$$\left(K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}\right)$$

- $\widehat{C}_{\mathrm{grav}}|\Psi\rangle = 0$

$$\widehat{C}_{\mathrm{grav}} = \frac{1}{16\pi l_{\mathrm{Pl}}^2\gamma^2}\left(\frac{96i(\mathrm{sgn}(p))}{8\pi\gamma l_{\mathrm{Pl}}^2}\widehat{F}\widehat{\frac{1}{\bar{\mu}^3}}\widehat{EE} + 2\gamma^2\Lambda\widehat{V}\right)$$

We expand states |ψ> in eigenstates |v>

$\Psi(v) = \langle\Psi|v\rangle$ and obtain the difference equation

$$|v+4||\,|v+3| - |v+5|\,|\Psi(v+4) - \{2|v||\,|v-1| - |v+1|\,| - \frac{128\pi}{81}\gamma^3\frac{l_P^2}{K^2}\Lambda|v|\Big\}\Psi(v) + |v-4||\,|v-5| - |v-3|\,|\Psi(v-4) = 0$$

# The behaviour of the wave fuction in the large volume limit

# Large volume limit

**Equi-area discretisation**

$$|V_{\mu+5\mu_0} - V_{\mu+3\mu_0}|\,\Psi(\mu+4\mu_0) - \left(2\,|V_{\mu+\mu_0} - V_{\mu-\mu_0}| - \frac{16\pi\gamma^3 l_{\rm Pl}^2 \mu_0^3}{3}\Lambda V_\mu\right)\Psi(\mu) + |V_{\mu-3\mu_0} - V_{\mu-5\mu_0}|\,\Psi(\mu-4\mu_0) = 0$$

⇒ WDW equation

(continuum limit)

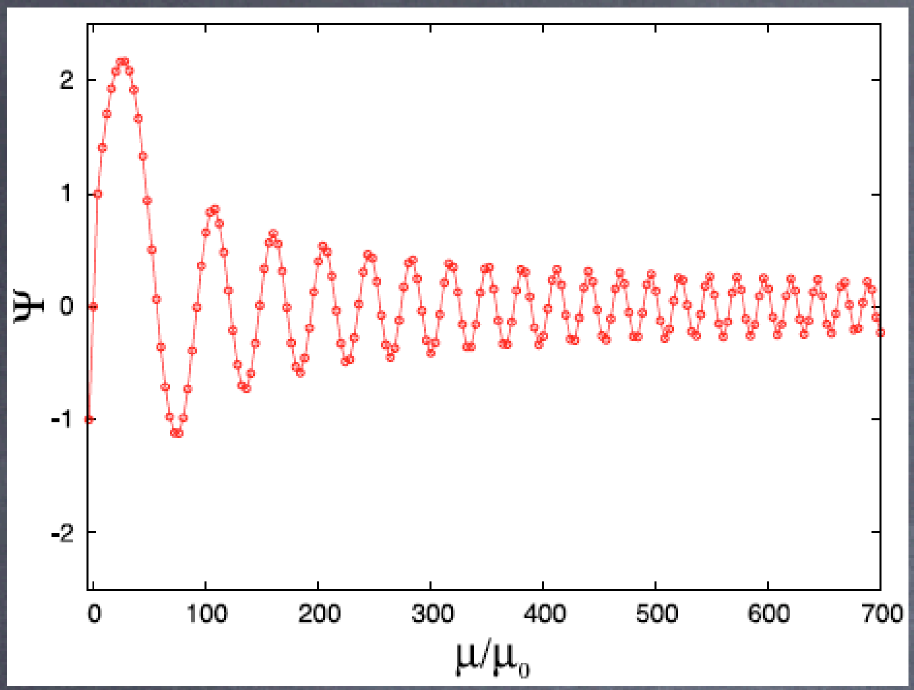
$$\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\left(\sqrt{\mu}\Psi(\mu)\right) + \frac{\pi\gamma^3 l_{\rm Pl}^2}{9}\mu^{3/2}\Lambda\Psi(\mu) = 0$$

the general solution : $\Psi(\mu) = \mu^{-\frac{1}{2}}\left[C_1 \mathrm{Ai}\left(-\alpha_1^{\frac{1}{3}}\mu\right) + C_2 \mathrm{Bi}\left(-\alpha_1^{\frac{1}{3}}\mu\right)\right]$

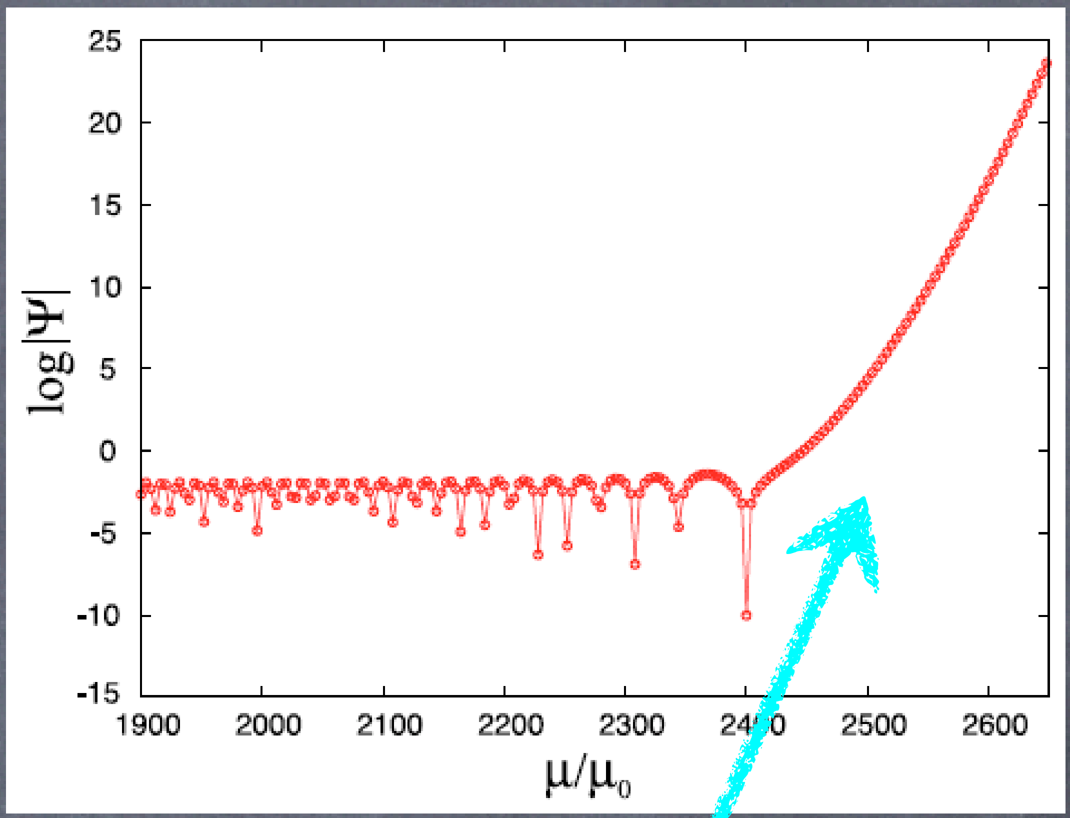
This is also the case in the equi-volume discretisation

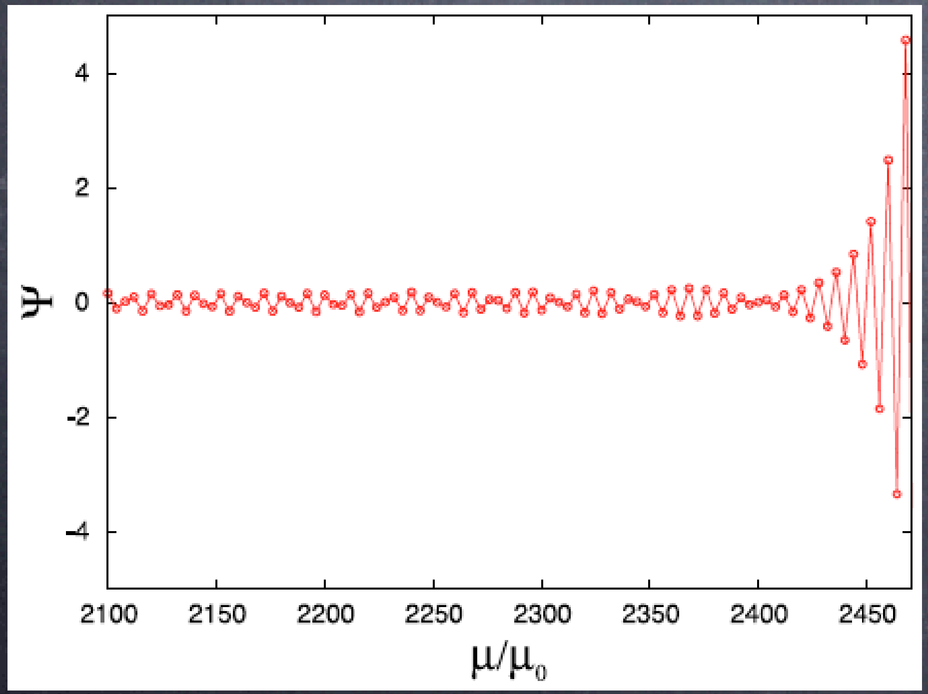
To see the large-volume behaviour of the wave function, we numerically solve the difference equation

# Equi-area discretisation




$0 \leqq \mu/\mu_0 \leqq 700$



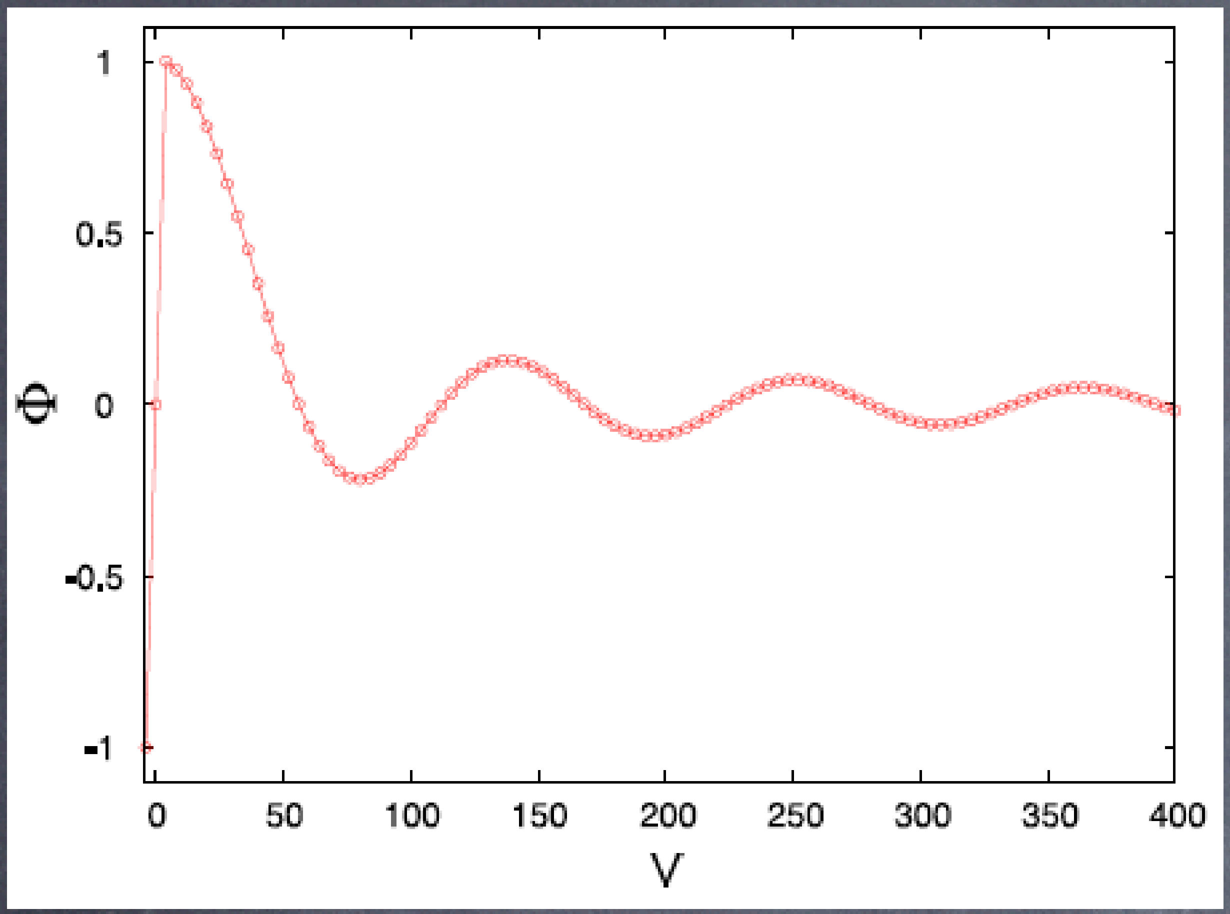




$2100 \leqq \mu/\mu_0 \leqq 2450$

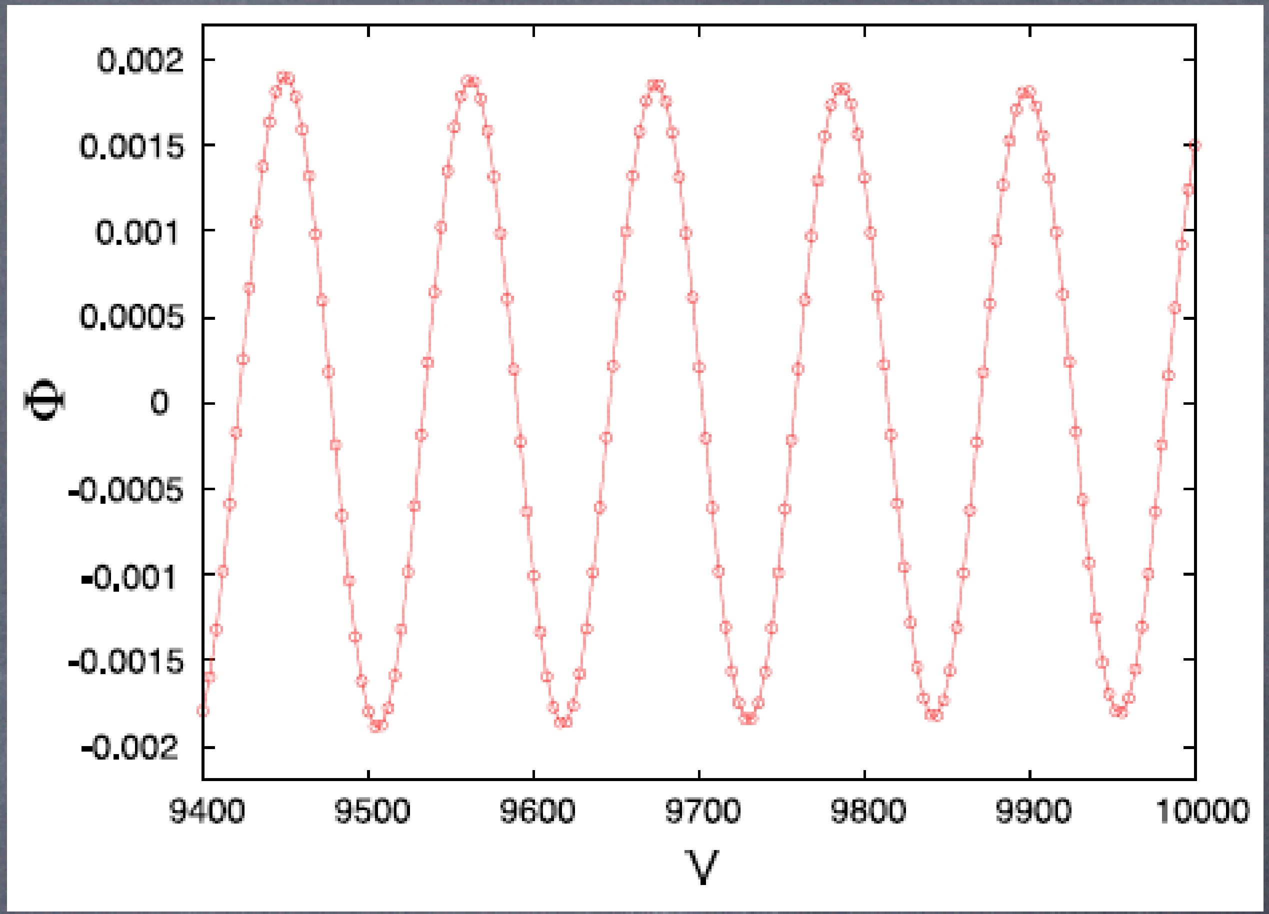
inconsistent with solutions of WDW eq.

The large volume limit problem

# Equi-volume discretisation




0≦v≦400




9400≦v≦10000

The large volume limit problem is resolved!

# Absence of
# the initial singularity

# Absence of the initial singularity

Bojowald (2001)

In LQC,

the inverse scale factor is bounded from above

→the physical quantities is bounded from above

We ONLY ask if the wave function can be uniquely extended beyond the classical singularity

# Operator orderings

Type (a) $\widehat{C}^{(a)}_{\text{grav}} = \frac{1}{16\pi l_{\text{Pl}}^2\gamma^2}\left(\frac{96i(\text{sgn}(p))}{8\pi\gamma l_{\text{Pl}}^2}\widehat{F}\widehat{\frac{1}{\bar{\mu}^3}}\widehat{EE} + 2\gamma^2\Lambda\widehat{V}\right)$

Type (b) $\widehat{C}^{(b)}_{\text{grav}} = \frac{1}{16\pi l_{\text{Pl}}^2\gamma^2}\left(\frac{96i(\text{sgn}(p))}{8\pi\gamma l_{\text{Pl}}^2}\widehat{\frac{1}{\bar{\mu}^3}}\widehat{F}\widehat{EE} + 2\gamma^2\Lambda\widehat{V}\right)$

Type (c) $\widehat{C}^{(c)}_{\text{grav}} = \frac{1}{16\pi l_{\text{Pl}}^2\gamma^2}\left(\frac{96i(\text{sgn}(p))}{8\pi\gamma l_{\text{Pl}}^2}\widehat{EE}\widehat{\frac{1}{\bar{\mu}^3}}\widehat{F} + 2\gamma^2\Lambda\widehat{V}\right)$

Type (d) $\widehat{C}^{(d)}_{\text{grav}} = \frac{1}{16\pi l_{\text{Pl}}^2\gamma^2}\left(\frac{96i(\text{sgn}(p))}{8\pi\gamma l_{\text{Pl}}^2}\widehat{EE}\widehat{F}\widehat{\frac{1}{\bar{\mu}^3}} + 2\gamma^2\Lambda\widehat{V}\right)$

$\left(\widehat{F} = \widehat{\sin^2\frac{\bar{\mu}c}{2}\cos^2\frac{\bar{\mu}c}{2}},\ \widehat{EE} = \widehat{\sin\left(\frac{\bar{\mu}c}{2}\right)V\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)V\sin\left(\frac{\bar{\mu}c}{2}\right)}\right)$

# Difference equations

Type (a)

$$|v+4|\,||v+3|-|v+5||\,\Phi(v+4)-\Big\{2|v|\,||v-1|-|v+1|| -\frac{128\pi}{81}\gamma^3\frac{l_{\rm Pl}^2}{K^2}\Lambda|v|\Big\}\Phi(v)+|v-4|\,||v-5|-|v-3||\,\Phi(v-4)=0$$

Type (b)

$$|v|\,||v+5|-|v+3||\,\Phi(v+4)-\Big\{2|v|\,||v+1|-|v-1|| -\frac{128\pi}{81}\gamma^3\frac{l_{\rm Pl}^2}{K^2}\Lambda|v|\Big\}\Phi(v)+|v|\,||v-3|-|v-5||\,\Phi(v-4)=0$$

Type (c)

$$|v|\,||v+1|-|v-1||\,\Phi(v+4)-\Big\{2|v|\,||v+1|-|v-1|| -\frac{128\pi}{81}\gamma^3\frac{l_{\rm Pl}^2}{K^2}\Lambda|v|\Big\}\Phi(v)+|v|\,||v+1|-|v-1||\,\Phi(v-4)=0$$

Type (d)

$$|v+4|\,||v+1|-|v-1||\,\Phi(v+4)-\Big\{2|v|\,||v+1|-|v-1|| -\frac{128\pi}{81}\gamma^3\frac{l_{\rm Pl}^2}{K^2}\Lambda|v|\Big\}\Phi(v)+|v-4|\,||v+1|-|v-1||\,\Phi(v-4)=0$$

We discuss the absence of the initial singularity in the equi-volume discretisation

# Type (a)

$$|v+4|\,||v+3|-|v+5||\,\Phi(v+4)-\{2|v|\,||v-1|-|v+1||$$
$$-\frac{128\pi}{81}\gamma^3\frac{l_{\rm Pl}^2}{K^2}\Lambda|v|\Big\}\,\Phi(v)+|v-4|\,||v-5|-|v-3||\,\Phi(v-4)=0$$

v = 4 $$16\Phi(8)-(16-4\tilde{\Lambda})\Phi(4)+0\times\Phi(0)=0$$

v = 0 $$8\Phi(4)-0\times\Phi(0)+8\Phi(-4)=0$$

v = -4 $$0\times\Phi(0)-(16-4\tilde{\Lambda})\Phi(-4)+16\Phi(-8)=0$$

⇨ We can solve the difference equation through the singularity v=0 and determine all ψ(v)

In this sense

"the initial singularity is avoided"

# Type (b)

$$|v|\,\big||v+5|-|v+3|\big|\,\Phi(v+4)-\Big\{2|v|\,\big||v+1|-|v-1|\big|$$
$$-\frac{128\pi}{81}\gamma^3\frac{l_{\mathrm{Pl}}^2}{K^2}\Lambda|v|\Big\}\,\Phi(v)+|v|\,\big||v-3|-|v-5|\big|\,\Phi(v-4)=0$$

v = 4 $\quad 8\Phi(8)-(16-4\tilde{\Lambda})\Phi(4)+0\times\underline{\Phi(0)}=0$

?

v = 0 $\quad 0\times\Phi(4)-0\times\underline{\Phi(0)}+0\times\underline{\Phi(-4)}=0$

? ?

v = -4 $\quad 0\times\underline{\Phi(0)}-(16-4\tilde{\Lambda})\underline{\Phi(-4)}+8\underline{\Phi(-8)}=0$

? ? ?

In this case
the initial singularity can NOT be avoided

# Absence of the initial singularity

| | Operator orderings | Absence or not? |
|---|---|---|
| Type (a) | $\hat{C}^{(a)}_{\mathrm{grav}} = \frac{24i(\mathrm{sgn}(p))}{8\pi\gamma l_P^2}\hat{F}\widehat{\frac{1}{\bar{\mu}^3}}\widehat{EE} + 2\gamma^2\hat{V}\Lambda$ | ○ |
| Type (b) | $\hat{C}^{(b)}_{\mathrm{grav}} = \frac{24i(\mathrm{sgn}(p))}{8\pi\gamma l_P^2}\widehat{\frac{1}{\bar{\mu}^3}}\hat{F}\widehat{EE} + 2\gamma^2\hat{V}\Lambda$ | × |
| Type (c) | $\hat{C}^{(c)}_{\mathrm{grav}} = \frac{24i(\mathrm{sgn}(p))}{8\pi\gamma l_P^2}\widehat{EE}\widehat{\frac{1}{\bar{\mu}^3}}\hat{F} + 2\gamma^2\hat{V}\Lambda$ | × |
| Type (d) | $\hat{C}^{(d)}_{\mathrm{grav}} = \frac{24i(\mathrm{sgn}(p))}{8\pi\gamma l_P^2}\widehat{EE}\hat{F}\widehat{\frac{1}{\bar{\mu}^3}} + 2\gamma^2\hat{V}\Lambda$ | × |

# Summary

- We discuss the singularity avoidance of the flat FRW universe with the cosmological constant in loop quantum cosmology

- The choice of the discretisation is crucial when we consider the large volume limit

  →In equi-area discretisation there arises the large volume problem
  In equi-volume discretisation this problem is resolved

- Absence of the initial singularity strongly depends on the operator orderings of quantum Hamiltonian operator

  →The requirement for the absence singles out a very small class of orderings

**Thank you for your attention!**

# Appendix

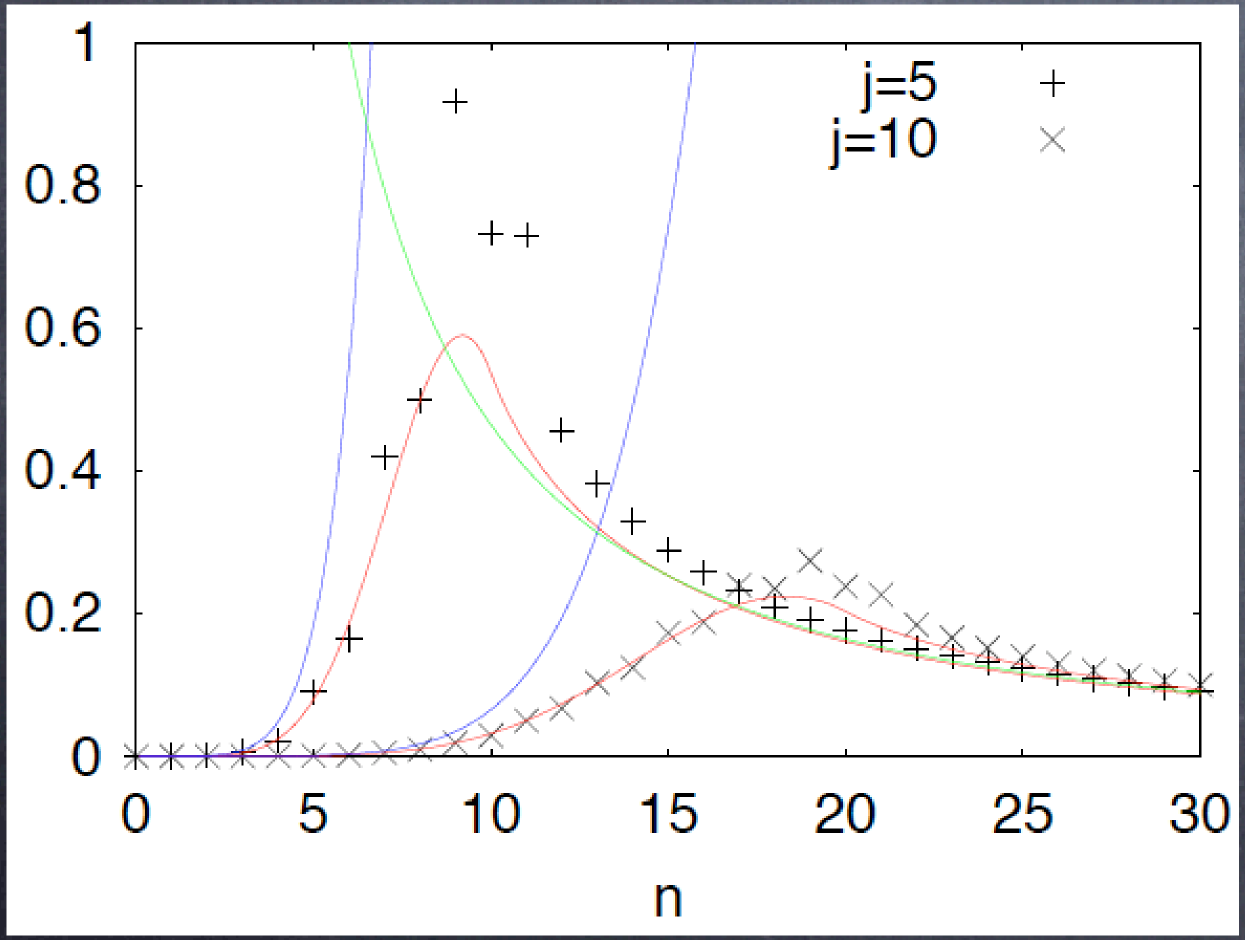
# Matter Hamiltonian

$$\hat{H}_{matter}|v=0\rangle = 0$$

ex) Massless scalar field

$$H_{matter} = \frac{p_\phi^2}{|p|^{3/2}} \left( = \frac{p_\phi^2}{a^3} \right)$$

Inverse triad operator : $\widehat{\frac{1}{p^{3/2}}}$




(M. Bojowld; "Loop quantum cosmology"; Living Rev.Rel.8:11,2005)

# Hamiltonian constraint

Hamiltonian (gravitational part)

$$H_{\rm grav} = \frac{1}{16\pi G}\int_\nu d^3x N\sqrt{h}[K_{ab}K^{ab} - K^2 + {}^{(3)}R]$$

rewritten by Ashtekar variables
in flat FRW spacetime

$$\begin{aligned} H_{\rm grav} &= \frac{1}{16\pi G}\int_\nu d^3x \frac{N}{\sqrt{|\det E_i^a|}}[\epsilon_{ijk}F_{ab}^i E^{aj}E^{bk} - 2(1+\gamma^2)K_{[a}^i K_{b]}^j E_i^a E_j^b] \\ &= -\frac{1}{16\pi G\gamma^2}\int_\nu d^3x \frac{N}{\sqrt{|\det E_i^a|}}\epsilon_{ijk}F_{ab}^i E^{aj}E^{bk} \end{aligned}$$

(+ Cosmological constant)

$$H_{\rm grav}^{CC} = \frac{1}{16\pi G\gamma^2}\int_\nu d^3x \left(\frac{-N}{\sqrt{|\det E_i^a|}}\epsilon_{ijk}F_{ab}^i E^{aj}E^{bk} + 2\gamma^2\sqrt{|\det E_i^a|\Lambda}\right)$$

# Loop quantization

We consider quantization of Hamiltonian constraint

$$H_{\rm grav} = -\frac{1}{16\pi G\gamma^2}\int_\nu d^3x \frac{N}{\sqrt{|\det E_i^a|}}\epsilon_{ijk}F_{ab}^i E^{aj}E^{bk}$$

In LQC (LQG)

We rewrite Hamiltonian constraint in the following form

(we need to use holonomy to obtain well-defined operator)

$$\epsilon_{ijk}\tau^i \frac{E^{aj}E^{bk}}{\sqrt{|\det E_i^a|}} = -\frac{2\mathrm{sgn}(p)}{\gamma\kappa V_0^{\frac{1}{3}}}\tilde{\epsilon}^{abc}\tilde{w}_a^i(h_i\{(h_i)^{-1}, V\})$$

$$\tau_i F_{ab}^i = \lim_{\mathrm{Area}\to 0}\left(\frac{h_\alpha - 1}{V_0^{\frac{2}{3}}}\right)\tilde{w}_a^i \tilde{w}_b^j$$

# Quantum Hamiltonian operator

Thus, we have quantum Hamiltonian operator

$$H_{\text{grav}} = \frac{1}{16\pi G\gamma^2}\int d^3x C_{\text{grav}}, \quad \text{where}$$

$$C_{\text{grav}} = -\frac{4\text{sgn}(p)}{8\pi\gamma\bar{\mu}^3 G}\sum_{ijk}\epsilon^{ijk}\text{Tr}[\underbrace{h_i h_j (h_i)^{-1}(h_j)^{-1}}_{(=F)}\underbrace{h_k\{(h_k)^{-1}, V\}}_{(=EE/\sqrt{\det E})}]$$

Quantization

$$\hat{C}_{\text{grav}} = \frac{24i(\text{sgn}p)}{8\pi\gamma\bar{\mu}^3 l_{Pl}^2}\sin^2(\bar{\mu}c)\left[\sin\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\sin\left(\frac{\bar{\mu}c}{2}\right)\right]$$