The complete solution of the conformastat electrovacuum problem

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Dedicated to the memory of S Brian Edgar

Conformastationary and conformastat spacetimes

Stationary spacetime $(\mathcal{M}, g_{\mu\nu})$: local coordinates $\{t, x^a\}$ $\exists$;

$$ds^2 = -e^{2U}(dt + A_a dx^a)^2 + e^{-2U}\widehat{h}_{ab}dx^a dx^b,$$

where U, A_a and $\widehat{h}_{ab}$ do not depend on t. U and A_a live on $(\Sigma_3, \widehat{h}_{ab})$

Static spacetime: $A_a = 0$.

Conformastationary spacetime: stationary spacetime where $(\Sigma_3, \widehat{h}_{ab})$ is conformally flat $\Leftrightarrow$ the York tensor density vanishes

$$Y_a{}^e = \hat{\eta}^{bce}\left(2\hat{\nabla}_c\widehat{R}_{ba} - \frac{1}{2}\widehat{h}_{ab}\hat{\nabla}_c\widehat{R}\right) = 0$$

$\widehat{R}_{ab}$, $\hat{\nabla}$ and $\hat{\eta}_{abc}$ relative to $\widehat{h}_{ab}$ $Y_{ae} = Y_{ea}$ and $Y_a{}^a = 0$.

Conformastat spacetimes: conformastationary spacetimes which are **static**.

Final result

Electrovacuum spacetime: solution of the Einstein-Maxwell field equations outside the sources.

All conformastat electrovacuum spacetimes **(inheriting)**

correspond to either

- the **Majumdar-Papapetrou class** of spacetimes
- the **static plane-symmetric** Einstein-Maxwell fields
- the **Bertotti-Robinson** conformally flat solution
- the non-extreme **Reissner-Nordström static** solution
- or the **hyperbolic** counterpart of the **Reissner-Nordström static** solution

$$ds^2 = -\frac{1}{V^2}dt^2 + V^2(dx^2 + dy^2 + dz^2) \qquad \forall \quad \hat{\nabla}^2 V = 0, \qquad \Phi = -e^{i\theta}\frac{1}{V}$$

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$$ds^2 = -\frac{(r+b)b}{r^2}d\tau^2 + \frac{r^2}{(r+b)b}dr^2 + r^2(d\vartheta^2 + d\varphi^2), \qquad \Phi = e^{i\theta}\frac{b}{r}$$

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$$ds^2 = -\sinh^2\left(\frac{z}{b}\right) d\tau^2 + dz^2 + b^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad \Phi = e^{i\theta}\cosh\left(\frac{z}{b}\right)$$

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$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) d\tau^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \Phi = e^{i\theta}\frac{Q}{r}$$

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$$ds^2 = -\left(-1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) d\tau^2 + \left(-1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sinh^2\vartheta d\varphi^2), \Phi = e^{i\theta}\frac{Q}{r}$$

Electrovacuum field equations

Inheriting Maxwell fields: $F_{\alpha\beta}$ for which $\mathcal{L}_{\partial_t}\boldsymbol{F} = 0$. The **Einstein-Maxwell equations outside the sources** imply $\exists$

- $\Phi(x^a)$ the **electromagnetic potential**
- $\mathcal{E}(x^a)$ the **Ernst potential**

; $H_a \equiv (\Re\mathcal{E} + \Phi\overline{\Phi})^{-1/2}\Phi_{,a}, \quad G_a \equiv 1/2(\Re\mathcal{E} + \Phi\overline{\Phi})^{-1}(\mathcal{E}_{,a} + 2\overline{\Phi}\Phi_{,a})$
satisfy

$$\widehat{R}_{ab} = G_a\overline{G}_b + \overline{G}_aG_b - (H_a\overline{H}_b + \overline{H}_aH_b),$$
$$\hat{\nabla}^aH_a + \frac{1}{2}\overline{G}\cdot H - \frac{3}{2}G\cdot H = 0,$$
$$\hat{\nabla}^aG_a - \overline{H}\cdot H - (G - \overline{G})\cdot G = 0.$$

Integrability conditions for the two potentials:

$$\mathrm{d}H = H\wedge\Re G \qquad \mathrm{d}G = G\wedge\overline{G} + \overline{H}\wedge H.$$

Metric determined by

$$e^{2U} = \Re\mathcal{E} + \Phi\overline{\Phi}, \quad \mathrm{d}A_{ab} = 2e^{-4U}\hat{\eta}_{abc}\Im G^c.$$

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Vacuum stationary and electro-magnetostatic cases

Vacuum Case: $\Phi = 0$, so that $H_a = 0$ and hence EM eqs. read

$$\widehat{R} = G_a \overline{G}_b + \overline{G}_a G_b,$$
$$\hat{\nabla}^a G_a - (G - \overline{G}) \cdot G = 0.$$

Integrability for potentials: $\mathrm{d}G = G \wedge \overline{G}$.

Static Case: $G_a - \overline{G}_a = 0$.
Then $G_a = U_{,a}$ and also $\overline{H}_a = e^{-2i\theta} H_a$ for constant θ.
Define $X_a \equiv e^{-i\theta} H_a = e^{-U} \Psi_{,a}$ where $\Psi \equiv e^{-i\theta}\Phi$ is real.
EM eqs. reduce to

$$\widehat{R} = 2(G_a G_b - X_a X_b),$$
$$\hat{\nabla}^a X_a - G \cdot X = 0,$$
$$\hat{\nabla}^a G_a - X \cdot X = 0.$$

Integrability for potentials: $\mathrm{d}H = H \wedge G, \quad \mathrm{d}G = 0.$

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Define $X_a \equiv e^{-i\theta} H_a = e^{-U} \Psi_{,a}$ where $\Psi \equiv e^{-i\theta}\Phi$ is real.
Define $\Sigma_a \equiv \frac{1}{2}(U_{,a} + jX_a)$.
(hypercomplex plane j; $j^2 = 1$ with conjugation $\breve{j} = -j$). EM eqs. read

$$\widehat{R}_{ab} = 4(\Sigma_a \breve{\Sigma}_b + \breve{\Sigma}_a \Sigma_b),$$
$$\hat{\nabla}^a \Sigma_a - (\Sigma - \breve{\Sigma}) \cdot \Sigma = 0.$$

Integrability for potentials: $\mathrm{d}\Sigma = \Sigma \wedge \breve{\Sigma}.$

Common framework

- Denote by ι both i and j, so that $\iota^2 = \pm 1$ accordingly, and by $\widetilde{\ }$ the general conjugation.
- Consider a "composed" vector field $\mathcal{Y}^a$ and a metric $\widehat{h}_{ab}$ that satisfy

$$\widehat{R}_{ab} = N(\mathcal{Y}_a \widetilde{\mathcal{Y}}_b + \widetilde{\mathcal{Y}}_a \mathcal{Y}_b)$$
$$\hat{\nabla}^a \mathcal{Y}_a - (\mathcal{Y} - \widetilde{\mathcal{Y}}) \cdot \mathcal{Y} = 0$$
$$\mathrm{d}\mathcal{Y} = \mathcal{Y} \wedge \widetilde{\mathcal{Y}}$$

- **Vacuum (stationary) case:** $N = 1$, $\mathcal{Y}_a (= G_a)$ complex
- **Static (electrovacuum) case:** $N = 4$, $\mathcal{Y}_a (= \Sigma_a)$ hypercomplex

Comformastationarity

Introducing the 1-form $L \equiv \star(\mathcal{Y} \wedge \widetilde{\mathcal{Y}})$ York $= 0 \Leftrightarrow$

$$(\mathcal{Y}_a - \widetilde{\mathcal{Y}}_a) L^e + \hat{\eta}^{bce} (\widetilde{\mathcal{Y}}_b \hat{\nabla}_c \mathcal{Y}_a + \mathcal{Y}_b \hat{\nabla}_c \widetilde{\mathcal{Y}}_a) - \frac{1}{2} \widehat{h}_{ab} \hat{\eta}^{bce} \hat{\nabla}_c (\mathcal{Y} \cdot \widetilde{\mathcal{Y}}) = 0.$$

Common framework

Solve the system. Sketch:

1. Case $\mathcal{Y} \cdot \mathcal{Y} = 0$:
   - **Vacuum:** $G \cdot G = 0 \Rightarrow$ flat. (Lukács, Perjés and Sebestyén, 1983)
   - **Static:** $\Sigma \cdot \Sigma = 0 \Rightarrow$ flat: trivial

Common framework

Solve the system. Sketch:

1. Case $\mathcal{Y} \cdot \mathcal{Y} = 0 \Rightarrow$ flat.
2. Case $L_a \neq 0$: Take the basis $\{L_a, \mathcal{Y}_a, \widetilde{\mathcal{Y}}_a\}$.
   - One of the $York = 0$ eqs. reads $L \wedge \mathrm{d}L = 0$. $\Rightarrow L = \iota\, \chi \mathrm{d}\varphi$.
   - $\mathrm{d}\mathcal{Y} = \mathcal{Y} \wedge \widetilde{\mathcal{Y}} \Rightarrow \exists$ composed σ;

$$\mathcal{Y} = \frac{1}{\sigma + \widetilde{\sigma}} \mathrm{d}\sigma$$

   Vacuum: $\sigma (= \mathcal{E}) = e^{2U} + i\Omega$ is the Ernst potential
   Static: $\sigma = \frac{1}{2}(e^{U} + j\Psi)$
   - Use the three potentials σ, $\widetilde{\sigma}$ and φ as coordinates. Write down the equations, and study the compatibility conditions, in particular for $N = 1$ and $N = 4$. Some long calculations show that **there are no solutions**.

Common framework

Solve the system. Sketch:

1. Case $\mathcal{Y} \cdot \mathcal{Y} = 0 \Rightarrow$ flat.
2. Case $L_a \neq 0$: **Is empty** $\Rightarrow L_a = 0$ necessarily, i.e. $\mathcal{Y} \parallel \widetilde{\mathcal{Y}}$

Theorem

Conformastationary vacuum spacetimes are always characterised by a functional relation between the potentials U and Ω. Perjés (1986a), Perjés(1986b)

Theorem

Conformastat electrovacuum spacetimes are always characterised by a functional relation between the potentials U and Ψ.

Conformastat electrovacuum: complete solution

Divergence equation for $\mathcal{Y}_a$ firstly fixes, for an arbitrary constant k,

$$e^{2U} = 1 - 2k\Psi + \Psi^2.$$

Can be rewritten in parametric form in terms of an auxiliary function V as

$$\begin{aligned}
k^2 = 1: &\quad \Psi = k - 1/V, &\quad e^{2U} &= V^{-2},\\
k^2 > 1: &\quad \Psi = k - \sqrt{k^2-1}\coth V, &\quad e^{2U} &= (k^2-1)\sinh^{-2} V,\\
k^2 < 1: &\quad \Psi = k - \sqrt{1-k^2}\cot V, &\quad e^{2U} &= (1-k^2)\sin^{-2} V.
\end{aligned}$$

Secondly, implies $\hat{\nabla}^2 V = 0$ in all cases. The Ricci equations read

$$\begin{aligned}
k^2 = 1: &\quad \widehat{R}_{ab} = 0,\\
k^2 > 1: &\quad \widehat{R}_{ab} = 2V_{,a}V_{,b},\\
k^2 < 1: &\quad \widehat{R}_{ab} = -2V_{,a}V_{,b}.
\end{aligned}$$

The remaining equations for $\widehat{h}_{ab}$ and V_a: York$=0$.

Conformastat electrovacuum: complete solution

$\widehat{h}_{ab}$ Ricci scalar	Ω_{AB}		
$\widehat{R} = 0$	any	**Majumdar-Papapetrou**	
$\widehat{R} > 0$	flat	Plane-symmetric field	
	spherical	**Bertotti-Robinson**	
		Reissner-Nordström	$M^2 - Q^2 > 0$
	hyperbolic	hyperbolic Reissner-Nordström	
$\widehat{R} < 0$	$\Rightarrow$spherical	**Bertotti-Robinson**	
		Reissner-Nordström	$M^2 - Q^2 < 0$

Corollary 1:

Improved (local) characterisation of Majumdar-Papapetrou

Before: static electrovacuum spacetime with **flat** $\widehat{h}_{ab}$ $(\widehat{R}_{ab} = 0)$
New:
static electrovacuum spacetime with **conformally flat** $\widehat{h}_{ab}$ and $\widehat{R} = 0$

Conformastat electrovacuum: complete solution

$\widehat{h}_{ab}$ Ricci scalar	Ω_{AB}		
$\widehat{R} = 0$	any	**Majumdar-Papapetrou**	
$\widehat{R} > 0$	flat	Plane-symmetric field	
	spherical	**Bertotti-Robinson**	
		Reissner-Nordström	$M^2 - Q^2 > 0$
	hyperbolic	hyperbolic Reissner-Nordström	
$\widehat{R} < 0$	$\Rightarrow$spherical	**Bertotti-Robinson**	
		Reissner-Nordström	$M^2 - Q^2 < 0$

Corollary 2:

Global consideration

The conformastat electrovacuum asymptotically flat spacetimes are

- the AF **Majumdar-Papapetrou**
- and the **Reissner-Nordström** static exterior